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Abstract

We analyze a search game in which multiple agents search an area (an island) for a hidden prize of known value. In every period until discovery, the rivals decide where and how much of the unsearched island to explore. The game ends when a player or players discover the prize. If one player discovers the prize on their own, they alone enjoy the spoils. Players have a per-period discount factor and costs proportional to how much they search. We compare two cases when: (i) the search of rivals (past and present) is observable to all; and (ii) players cannot observe others' previous and current search. We show that welfare in the unique SMPE with observability is always (weakly) higher than in the case without observability. However, we show that there is a self-enforcing mechanism without observability in which a third party ex ante allocates search zones to each of the players that ensures the same outcome is achievable as with observability. Our results have implications for the design of search games (patents, prizes, information sharing, and so on) by regulators.

Keywords: search, uncertainty, regulatory design.
JEL classifications: D21, O32.

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1 Introduction

It can seem at times that we spend our whole lives searching for things: lost keys, that pesky cell phone, your children. While possibly less important, firms also invest significant resources in search, be it exploring for a new mineral deposit, product or market opportunity. In these and other business contexts, strategic interaction between firms is often of vital importance, with rivals competing with one another to make the discovery first and to claim the ensuing rewards. Moreover, the firms themselves might be cognisant that they could be looking for the prize in the same location, just like prospectors inadvertently uncovering the same deposit or pharmaceutical companies unknowingly pursuing similar research programs. Joint discovery has the potential pitfall that the prize will be largely squandered through excessive competition (or conflict). This possibility, in turn, will affect the search strategies employed by the parties. Search – and the search environment – is the topic of this paper.

We develop a model in which n firms or players undertake costly search for a prize (reward or treasure) of known value that is hidden somewhere on an island of finite size. It is assumed that there is an equal probability that the reward will be located at any given point on the island. In each period, the players make their search decisions simultaneously. The search process ends in the period in which the treasure is found. If a firm makes the discovery on their own, it enjoys the spoils by itself. If more than one of the rivals simultaneously discover the prize (by successfully searching the same area in a given period), we assume that the treasure is destroyed, dissipating any possible rents. This assumption is akin to Bertrand competition, in which tough competition results in low prices and profit margins. The dissipation could also reflect the costs of a protracted legal battle between the potential claimants.[1]

One of the key aspects we investigate here is the role of observability of rivals' search. To do so we compare two situations. Firstly, as a benchmark for our analysis, we allow players to be able to observe their rivals' past and present

[1]For example, the competition between jet developers Lockheed and Douglas in the 1960s dissipated much of the potential rents for both firms. For more detail, see *The Economist (1985)*; and Chatterjee and Evans (2004). For examples of simultaneous discoveries in science see Merton (1973).

search activities. In many situations it is possible, after-the-fact, to discern what a rival did. Observability of past actions allows a firm to avoid duplicating past unsuccessful search. If it is possible to observe current search activity, firms will be able to avoid searching the same area at the same time. Observability in this sense helps the firms avoid duplication of both past unsuccessful search and current efforts. This assumption follows the duopoly models of de Roos et al (2018) and the coordinated search case studied in Matros and Smirnov (2016).

Secondly, we assume that players cannot see any of the past (or current) search conducted by their rivals. Non-observability can apply to various situations, including small scale gold prospecting, paparazzi looking for an elusive celebrity and the mining of Bitcoin. This means that search plans in any periods cannot be conditioned on past actions of other players. Moreover, duplication – that is firms searching the same area in a given period – is possible. As players do not observe the locations that have been previously searched by other players, using the terminology of Mailath and Samuelson (2006), we can think of this as a stochastic game with private monitoring in a dynamic setting. As discussed further below, Fershtman and Rubinstein (1997) and Chen et al (2015) theoretically examine search when rivals' actions are unobservable.

We focus on the symmetric Markov Perfect equilibrium (SMPE). With non-observability, the equilibria mimic open-loop equilibria, given firms inability to react to the past search of their rivals. Likewise, search with observability naturally lends itself to closed-loop equilibria, as analyzed in Matros and Smirnov (2016) and de Roos et al (2018).

Several of the key results of the paper are as follows. Firstly, when firms cannot observe the past search of their rivals, the expected payoffs to the firms in the symmetric equilibrium cannot exceed those in the SMPE with observability. When rivals' search is unobservable, duplication of search effort dissipates the expected return; in fact, it is possible that all potential rents are dissipated and the expected value to each player is zero. However, with the assistance of a third party such as regulator or industry association, the outcome in the case without observability can replicate the payoffs in the observable SMPE. This mechanism works as follows. The third party divides the whole island into equal portions or zones and allocates each of searchers one of these portions. Each party is then expected to only search in their allocated zone. Importantly, the mechanism is credible in that each of the parties will find it in their interest not to search in any of their rival's allocated zones. Crucial to this result is that search in any particular area is equally likely to be successful, but searching in a rival's zone entails a positive probability of duplication (squandering the benefit of any potential find). This finding has implications for the design and regulation of environments in which search is naturally unobservable.

Search has been studied by others in complementary settings to ours. As noted above, we draw heavily on the observable-search framework used in Matros and Smirnov (2016) and de Roos et al (2018). In their duopoly model, Matros and Smirnov (2016) focus on duplication of search, particularly when current search cannot be coordinated.[2] de Roos et al (2018) analyze the welfare implications of monopoly and duopoly search, allowing for coordination in the current period. We extend their analysis in two key ways. Firstly, we extend their model with observable actions to n players. Secondly, as noted above, here we allow for environments in which rivals' search is secret. This allows for a direct comparison of search with and without observability. It also raises the question as to what sort of mechanisms could be utilized to avoid costly duplication. Here the mechanism suggested is a third party (or principal/market designer) who can allocate exclusive search zones.

Our use of exclusive search zones to overcome duplication when there is no observability has similarities to the mechanism employed by Chen et al (2015), who investigates how rival treasure hunters should share information in a command-control economy. In their model, individual agents each have some information about where the treasure could be, but this information can differ between agents. The key idea in their paper is that in this context sharing information improves welfare.[3] In order to illicit the agents' information Chen et al (2015) propose the following mechanism: agents reveal their information sets as to where the treasure could be located; the principal ascertains the intersection of these information

[2]Inefficient duplication arises also in the dynamic R&D search game of Chatterjee and Evans (2004).

[3]They are also concerned about the equity of the mechanism, in that agents with more valuable information should not be surpassed in their probability of winning by those with less valuable information relative to search without the sharing rule.

sets and allocates search zones to each agent from the intersection; and finally, agents search for the treasure, but can only search within their allocated zone. We discuss further the implications of this paper in Section 5, however, one key difference between their model and ours is that in Chen et al (2015) the mechanism can be enforced, whereas we do not assume that this need be the case, widening the applicability of our mechanism.

The model of Fershtman and Rubinstein (1997) is also closely related to our study here. As in our model, rival search is unobservable, so there is always the possibility of costly duplication. One key difference between their model and ours, however, is that in Fershtman and Rubinstein (1997) each player chooses their per-period search intensity ex ante, whereas here each player can reevaluate how much to search after every period.

Our work extends the previous research on private-good search. These models mostly deal with situations that are either static or involve complete information.[4] A notable exception is the seminal dynamic-investment model of Reinganum (1981). In that model, two rivals invest to accumulate knowledge over time with the aim to successfully innovate, which occurs with an exponential probability; Reinganum (1982) uses a similar framework to show how the availability of patent protection can accelerate development of the innovation. As well as focusing on the case in which past search is not observable and there is the possibility of duplication, a contribution of our paper is that we analyze equilibrium search without relying on an exponential success function and a stationary environment; that is, in our model the players' choices regarding effort depend on the remaining unsearched island size.

There is also a large literature on investment dynamics in relation to public goods.[5] Perhaps the key paper for us in this literature is Bonatti and Hörner (2011), who examine a dynamic moral-hazard-in-teams problem, in which agent effort in each period increases the prospects that a project will be successful. Search ceases in the period in which a successful project is discovered, and agents

[4]See, for example, Loury (1979), Dasgupta and Stiglitz (1980a,b) and Lee and Wilde (1980), and Reinganum (1989) and Long (2010) for a survey of the literature.

[5]See for instance, Admati and Perry (1991), Marx and Matthews (2000), Lockwood and Thomas (2002), Compte and Jehiel (2004), Yildirim (2004, 2006), Matthews (2013), Battaglini et al. (2014) and Georgiadis (2015).

choose their levels of effort in each period. Upon success, each agent receives the payoff from the prize; this public good nature of their model differs from ours in which the prize is enjoyed by the individual discoverer alone. Their results on the observability of actions in a public-good search game provide a nice counterpoint to our private-good search environment. We find that observability improves the expected outcome by helping avoid duplication. In contrast, in their moral-hazard-in-teams model Bonatti and Hörner (2011) find that better monitoring (better observability) can induce greater delays in discovery, as compared to when agents' effort are unobservable.

2 The model

The setup. Assume that n players (firms) search for a treasure hidden somewhere on an island of size $x_1 \in (0, 7/8]$ over potentially an infinite number of periods; search ceases in the period in which the treasure is discovered.[6] Payoffs are discounted at a common factor $\delta < 1$. We make the simplifying assumption that there is an equal probability of the treasure being buried at any particular place, so there is no *a priori* advantage of searching one particular spot over another.[7]

Timing. Each period t comprises the following sequence of events. Firstly, if applicable, each firm recalls the details of their unsuccessful search efforts in terms of its size and location from previous periods. As a convention, we use the term 'area' to mean the size of land searched and the term 'region' to mean the location of the search. Let $x_{i,t}$ be the unsearched area of the island at the beginning of period t for player i. Secondly, each player i simultaneously plans an area $I_{i,t} \in [0, x_t]$ and a region to search at a cost of $I_{i,t}$.[8] Thirdly, search plans

[6] The restriction here on the island size is for expositional purposes, as it ensures that search is completed in at most two periods. Previously de Roos et al (2018) have considered equilibria without such a restriction when $n = 2$. Our focus here is on a comparison between the observable and non-observable search outcomes, and this helps avoid some issues of multiple equilibria that are peripheral to the main question at hand.

[7] Matros and Smirnov (2016) consider the case when the distribution is non-uniform. Also see the discussion in Chen et al (2015). For the sake of clarity of exposition, we do not consider this issue here.

[8] Non-linear costs are discussed in Matros and Smirnov (2011). As noted there, their inclusion would not alter the key intuition of our results.

are implemented. Following this, if the treasure is found, the game ends and the payoffs (if there are any) are realized. As noted above, if one firm discovers the prize on their own, they receive the full prize $R = 1$.[9] On the other hand, if two or more firms discover the prize simultaneously (which would arise if they successfully searched the same region) the potential prize is destroyed, and each party receives a payoff of zero (while still incurring their search costs). As noted, the assumption here is that joint discovery unleashes such fierce competition that any potential rents are dissipated (akin to homogenous-good Bertrand competition).

Information. We are interested in comparing two contrasting search environments. In both cases every firm knows exactly what they themselves have done in the past – where and how much they have unsuccessfully searched. However, what a firm knows about the previous search of their rivals could be different in each of the two cases. In the first search environment, each firm can observe what their rivals have been up to and where they are currently searching. Mining companies might be able to observe the exploration of rival firms, either from the location of physical assets or from documents submitted to a public registry. Similarly, pharmaceutical companies and other research units might be able to ascertain the past and present efforts of rival outfits from public documents and publications, the type of researchers they hire or mandatory reporting to co-funding agencies. Moreover, corporate espionage, while not legal, might help a firm know what its rivals have been up to and what they are currently doing. To capture this, we first assume that each player can observe all search by its rivals, including their current search plans. This means that firms will avoid searching areas previously fossicked, but can also effectively coordinate their current search plans so as to avoid duplication. We consider these situations in Section 3.

On the other hand, sometimes firms are unable to ascertain how much and where their rivals search. All they can really know at the start of a period is that no one found the treasure. This assumption applies to many different search examples. Paparazzi might not be able to observe where photographers from

[9]As players are risk neutral, in the case of random returns R could be replaced by $E[R]$, requiring no other changes to the model. Note also that previously Matros and Smirnov (2016) and de Roos et al (2018) considered equilibria when $R \neq 1$. Relaxing this assumption does not add anything beyond their analysis. We do not consider either of these issues here so as to be as parsimonious as possible.

a rival publication have been looking for an elusive celebrity. Enemy bands of pirates might be unable to see where rival marauders have trolled for lost treasure. Similarly, software companies might not be able to see the details of their rivals' unsuccessful attempts to develop a new product. These situations are captured in Section 4, in which rivals are unable to observe one another's search, both past and present.

Both search environments have been the subject of past research, with papers like Matros and Smirnov (2016) and de Roos et al (2018) analyzing the observable rival actions scenario, whilst Fershtman and Rubinstein (1997) and Chen et al (2015) focus on the case without observability. The setup in this paper allows for a direct comparison of the search behavior and welfare under both environments.

Equilibrium solution concept. We consider Markov strategies in which prior search influences current play only through its effect on the current unsearched area. A pure Markov strategy for player i is a time-invariant map $I_i : X \to X$, where $X = [0, x_1]$ and $I_i(x) \in [0, x]$. We restrict attention to symmetric equilibria, using *symmetric Markov perfect equilibrium* (SMPE) as the solution concept. Moreover, we focus on *non-trivial* SMPE; that is, equilibria in which a positive amount of search occurs somewhere along the equilibrium path. Reflecting our consideration of symmetric Markov strategies, we omit the time subscript and indicate the search intensity function for firm i as $I_{i,t} = I_i(x)$. Where the meaning is clear, we also omit the subscript i in describing the equilibrium search and value functions.[10]

3 Search with observability

In this section we analyze the equilibrium of the model with observability, generalizing de Roos et al (2018) for n players. As a convention, we use superscripts in parentheses to indicate the number of searching parties.

Let us first begin with the Bellman equation faced by player i. In this case:

[10] We define equilibria in terms of the functions I_i where $i = 1, \ldots, n$ that describe the area but not the region of search. An equilibrium is therefore consistent with a multiplicity of outcomes relating to the search region.

$$V(x) = \max_{I_i \in [0,x]} \left\{ -I_i + \frac{J - I_{-i}}{x} + \delta \left(1 - \frac{J}{x}\right) V(x - J) \right\}, \qquad (1)$$

where x is the area of the island that is unsearched before the current period, $V(x)$ is the value function for each player (we use the symmetry assumption here), $J = \min\{x, \sum_{i=1}^n I_i\}$ is the aggregate search this period by all players, and $I_{-i} = \min\{x, \sum_{j \neq i}^n I_j\}$ is the aggregate search this period by all players other than i.

In equation (1): the first term is player i's current search costs; the second term describes her expected value from a successful current search; and the last term is her expected discounted value from future search.

From this, we can give a complete characterization of all SMPE, as summarized in the following proposition.

Proposition 1. *Suppose $n \geq 2$ and $x \leq \frac{7}{8}$. Then, (a) a unique non-trivial SMPE exist as described below; and (b) all search is conducted within two periods. The search intensity of each firm is*

$$I^{(n)}(x) = \begin{cases} \frac{x}{n}, & \text{if} \quad x \leq 1 - \frac{\delta}{n}, \\ \frac{(1-x)(n-2\delta)+\delta}{2\delta n}, & \text{if} \quad 1 - \frac{\delta}{n} < x \leq \frac{7}{8}; \end{cases}$$

and the value function for each firm is

$$V^{(n)}(x) = \begin{cases} \frac{1-x}{n}, & \text{if} \quad x \leq 1 - \frac{\delta}{n}, \\ \frac{-(1-x)^2(4\delta+n(n-2))+2\delta(1-x)+\delta^2}{4\delta n x}, & \text{if} \quad 1 - \frac{\delta}{n} < x \leq \frac{7}{8}. \end{cases}$$

Proof. See Appendix A.

In a non-trivial SMPE, firms plan to search the island for at most two periods. If the island is sufficiently small ($x \leq 1 - \frac{\delta}{n}$), the players search the entire island in a single period. For $x > 1 - \frac{\delta}{n}$, the firms plan to search the island for at most two periods. In sum, the equilibrium search intensity $I^{(n)}(x)$ is a spline of degree one on the interval $[0, 7/8]$ with a possible single knot $1 - \frac{\delta}{n}$.

We characterize the equilibrium search intensity and the value function in this case with the following corollary, which follows directly from the previous proposition.

Corollary 1. *When $x \leq 7/8$ and $n \geq 2$, the equilibrium search intensity $I^{(n)}(x)$ is a unique, piece-wise linear, continuous and concave function, while the corresponding value function $V^{(n)}(x)$ is a unique, continuous, non-monotonic and quasiconvex function.*

Panel (a) of Figure 1 illustrates the equilibrium search intensity in the SMPE for each of the rivals when $\delta = 0.9$ and $n = 2, 3$ and 4. Consider first the duopoly search model ($n = 2$). In this case search is concluded within two periods, and there is only one knot of the spline. There is a kink in the Figure at the knot, but search intensity is continuous. This is because searching for one period (which occurs for $x \leq 1 - \frac{\delta}{n}$) is equivalent to searching for two periods with all the search conducted in the first period. Note that equilibrium search intensity is a non-monotonic function of island size. For sufficiently small islands, the firms search the entire island in a single period, and search intensity is therefore increasing in island size for $x \leq 1 - \frac{\delta}{n}$. Beyond the knot, the intensity of search decreases with island size, reflecting the diminished probability of successful search in the first period. Note that when $n = 3$ and $n = 4$ the knot shifts further to the right; multi-period search is more difficult to sustain when there is a larger number of players.

Panel (b) of Figure 1 illustrates the value function for each of the rivals when $\delta = 0.9$ and $n = 2, 3$ and 4. Consider $n = 2$. The value function is monotonically decreasing with the size of the unsearched area because it is harder to find the treasure on a larger island. However, the value function is not smooth; the slope of the value function changes at knot $1 - \frac{\delta}{n}$, coinciding with the discontinuity of the slope of the equilibrium search intensity. The discontinuity relates to the inefficiency of rival search for $x > 1 - \frac{\delta}{n}$. Note also that when $n = 3$ and $n = 4$, this inefficiency is so big, that value function is not monotonic with island size.[11]

4 Search without observability

As noted previously, in many situations players cannot keep tabs on what their rivals are doing, or even what they have previously done. In this case, rival search

[11]de Roos et al (2018) analyses value non-monotonicity in the case of $n = 2$ players, which requires at least 3 periods of search and the island size larger than 7/8.

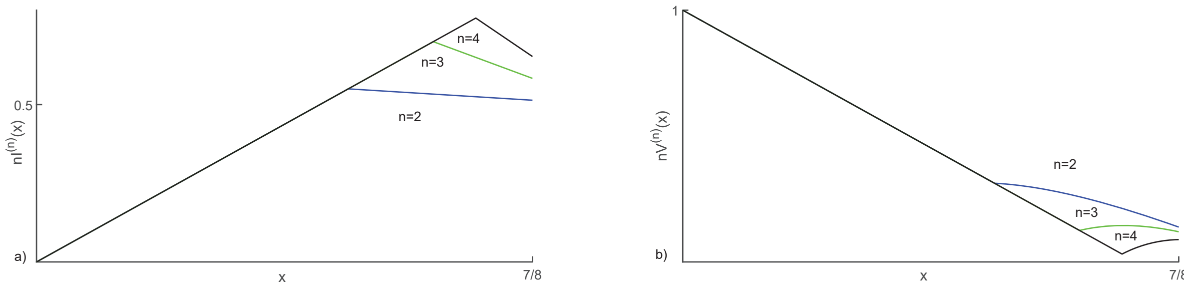

Figure 1: n-player search intensity and value when $\delta = 0.9$

is effectively unobservable. As we will show, this makes it difficult for the parties to avoid costly search duplication. One example we have in mind is pharmaceutical companies trying to develop a new product, such as a sleeping pill. Given the secrecy surrounding work in these laboratories, it is not possible to guarantee that there is no duplication by rivals.[12]

To explore this issue, first we examine decentralized search, when the players search independently without any third-party guidance or advice. Secondly, we allow a third party, such as a regulator, who can try to facilitate some sort of coordination in the hope of reducing duplicative search.

4.1 Decentralized and uncoordinated search

Let us consider the extreme case when players cannot observe past or present actions of their rivals and there is no coordination. The only information a player has is: where they previously unsuccessfully searched; and that no one has yet found the treasure. Given this, a player effectively randomly selects the region to search (from their unexplored portion of the island), and these selections are made independently between players, in a similar manner to Fershtman and Rubinstein (1997) and Matros and Smirnov (2016). In this case the Bellman equation faced by player i is

[12]For example, Eichler et al. (2013) discuss a documented case of duplication. For further examples see Chatterjee and Evans (2004).

$$V(x) = \max_{I_i \in [0,x]} \left\{ -I_i + \frac{I_i}{x}\left(1 - \frac{I}{x}\right)^{n-1} + \delta \left(1 - \frac{I_i}{x}\right)\left(1 - \frac{I}{x}\right)^{n-1} V(x - I_i) \right\}, \tag{2}$$

where x is the area of the island that is unsearched before the current period, $V(x)$ is the value function for each player (we use the symmetry assumption here), I_i the search this period by player i and I is the search this period by any other player j. In equation (2) the first term is player i's current search costs I_i. The second term describes her expected value from a successful current search, which is the probability that her search is successful but all of her rivals were not. Finally, the last term is her expected discounted value from future search, which is relevant when no one was successful in finding the prize in the current period.

We capture the equilibrium in this case in the following proposition.

Proposition 2. *In the unique SMPE* $I(x) = x - x^{n/(n-1)}$ *and* $V(x) \equiv 0$.

Proof. See Appendix A.

To get some intuition, let us consider the static problem where players can only search once. The problem in this case is

$$V(x) = \max_{I_i \in [0,x]} -I_i + \frac{I_i}{x}\left(1 - \frac{I}{x}\right)^{n-1}.$$

If the search by each other player is $I < x - x^{n/(n-1)}$, it is a dominant strategy for player i to choose $I_i(x) = x$. If, on the other hand, search by each other player is $I > x - x^{n/(n-1)}$, it is a dominant strategy for player i to opt $I_i(x) = 0$. There is a unique symmetric equilibrium in which each player searches $I(x) = x - x^{n/(n-1)}$, and receives a value of zero.

In the equilibrium to the dynamic game, the island has no value to either player for any x. To see why this is an equilibrium, note that if $V(x) = 0$ for all x, then (2) reduces to the static problem above, and the unique solution has a value of zero for all players. Because of the contraction property of the dynamic problem, this equilibrium is unique.

Note, not only do we have duplication in this case, the duplication here is so egregious that the whole value of the treasure is dissipated in equilibrium. Thus, random search gives rise to an extreme example of the tragedy of the commons. A feature of equilibrium is that the island is never searched completely and search may continue for arbitrarily many periods.

4.2 Third-party coordination

A question that follows immediately from the analysis in the previous section is 'can't the searching parties do better than this?' One of the key issues when players are unable to observe their rivals' search is duplication. This is not an issue when there is observability; rivals avoid costly duplication by effectively staying on 'their own turf' and keeping out of each other's way. Specifically, is it possible to implement some coordination mechanism that can help the players avoid costly duplicative search when actions are unobservable?

To explore this, let us first compare the expected payoffs in environments with and without observability. Specifically, the following lemma puts an upper bound on the equilibrium value when actions are not observable.

Lemma 1. *For any equilibrium derived in the environment with no observability the following inequality always holds:*

$$V_{nob}(x_1) \leq V^{(n)}(x_1). \tag{3}$$

Proof. See Appendix A.

This Lemma shows that the expected payoff in any symmetric equilibrium without observability cannot be more than the highest payoff in the equilibrium with observability, as analyzed in Section 3.

Let us show that the upper limit of payoffs ($V^{(n)}(x_1)$ from the equilibrium with observability) described in Lemma 1 can be achieved as an equilibrium. To frame this discussion, consider the situation when players can informally coordinate on how to divide the island in n equal zones so that, notionally, each player is allocated a zone that is 'their turf'. To do this, let us introduce a third-party regulator, whose only task is to allocate search zones to each player.[13] The idea here is that the regulator assigns each of the searchers a particular area that does not overlap with the allocated zone of any other player. The full island is covered by the allocated zones, and each player gets $1/n$-th of the total search area. This idea is summarized in the following definition.

Definition 1. *Zone z_i is the area assigned to player i prior to any search activities by a third-party regulator. All zones are of equal size and the sum of all zones covers without duplication the entire island x_1.*

Importantly, we assume that these allocated zones are not contractible in that there is no formal sanction for any player who strays and searches outside of their allocated zone; rather, to be effective it needs to be self-enforcing. The zones here could be allocated areas of research proposed by a research body. In this case, the allocation is suggestive rather than being formally enforceable, because a successful discovery made outside of one's zone would never be credibly punished. To effectively prevent duplication, each player must choose not to search outside of their zone.

Consider formally, the strategy of each player, after they have been allocated their zone. The strategy of each firm will entail a choice of how much of its own zone of the island, as well as how much of the other player' zone, to search each period. As noted, if any firm searches their rival's notionally allocated zone, there could be some duplication.

In the next proposition we show that the outcome observed in the SMPE in the game with observability can be supported as an equilibrium when there is no observability. Formally,

Proposition 3. *With third-party coordinated zones, in the SMPE,*

$$V_{nob}(x_1) = V^{(n)}(x_1).$$

Proof. See Appendix A.

[13] A regulator as such is not necessary. One or a group of active searchers could act as the coordinator in this case.

The intuition for this result is as follows. When the search requires covering relatively large areas in each period, it is not beneficial to search in the zone that has been 'flagged' to be the rival's zone. As a player is assumed to search his own zone randomly, searching in a rival's zone results in a positive probability of duplication. With relatively large search pattern the probability of duplication is high when searching in a rival's zone, which acts as a credible disincentive to move outside of one's own zone. On the other hand, when the search requires covering relatively small portions each period, it may be worth searching the area of the other player. The following condition guarantees no deviation

$$-\varepsilon + \frac{\varepsilon(x - nI^{(n)})}{x^2} \leq 0, \tag{4}$$

where the first term is the cost of searching the other player' area, while the second term is the expected benefit to the deviator who searches ε, while the 'incumbent' searches $I^{(n)}$ on his area x/n. In the proof of Proposition 3 we show that condition (4) is always satisfied.

5 Implications

Firms often like to keep things a secret. If they can, a firm will keep its rivals in the dark about its research efforts and its findings. But, as we know, what is in the private interest of an individual firm and what is best for social welfare need not be the same. Our search model is another example of this maxim. In this paper, we analyze the search strategies and payoffs with both observable and unobservable rival search. The assumption of unobservability is appropriate when parties undertake their search in secret, as is often the case when firms are developing products or exploring new market opportunities. Importantly, we show that the expected payoff cannot be higher with non-observability than it is with observable search. The key element driving the additional loss of surplus with unobservable actions is duplication, both in terms of searching areas previously examined, and players searching the same space currently.

We flagged an self-enforcing coordinating device in which each player is allocated a research zone. These zones would then corral each firm's search so as to avoid duplication. Even if they are not legally enforceable, the creation of zones coordinates the actions of independent agents so as to help avoid costly duplicative search, and achieve more efficient search. Our self-enforcing coordination mechanism complements the enforceable mechanism studied in Chen et al (2015). The need for a self-enforcing approach is relevant in many search environments. It might be difficult for a third party to sufficiently observe and verify where an agent searched to the degree to which any breach could be prosecuted. In traditional search environments, like prospecting and treasure hunts, parties might not be able to commit and the legal structure might be ineffective to prevent cheating. In the research and development application, while the experts themselves might be able to describe designated research regions for one another, these scientific nuances might be lost in court, leading to a classic incomplete contract. Given this, self-enforcing mechanisms are likely to be important in many search context. By relaxing the assumption of enforceability, we allow players to endogenously choose whether to adhere to the authority's proposed division of search areas.

Finally, we have considered here two ends of the spectrum with unobservable search. If the parties' actions are uncoordinated, the duplication is so egregious that all rents are dissipated. On the other hand, third-party zoning allows the unobservable outcome to replicate that with observability. But it is also possible to have every outcome, in terms of equilibrium value and efficiency, in between these two extremes if the third party allocates zones that are not necessarily exclusive. Moreover, the parties themselves, while aware that their territories are not necessarily exclusive, might not know exactly where the intersection of the zones is. This will lead to an outcome somewhere in the middle of the two extreme cases.

Appendix A

We begin with some further notation. When convenient we omit: firm and time subindexes; and superscripts indicating how many firms searching.

It proves convenient to introduce the following function:

$$\Psi(x) \equiv xV(x). \tag{5}$$

From equation (1) and definition (5), it follows that $\Psi(x) \geq 0$ for any x and

$$\Psi(x) = \max_{I_i \in [0,x]} \{J - I_{-i} - xI_i + \delta\Psi(x - J)\}. \tag{6}$$

Define the value-iteration operator B as follows:

$$(B\Psi)(x) \equiv \max_{I_i \in [0,x]} \{J - I_{-i} - xI_i + \delta\Psi(x - J)\}. \tag{7}$$

The following Lemma was proved in Matros and Smirnov (2016) for $n = 2$. We generalize this result for any n.

Lemma 2. *There exists finite T such that:*
(a) any non-trivial SMPE involves $J_t > 0$ for $0 \leq t \leq T$, and $x_{T+1} = 0$; and
(b) all SMPE can be obtained in T steps. Specifically,

$$\Psi_0 \equiv 0, \qquad \Psi_s \equiv B\Psi_{s-1} \qquad \textit{for} \quad s = 1, 2, \ldots, T, \qquad \textit{and} \qquad \Psi = \Psi_T.$$

Proof of Lemma 2

Suppose (a) is not satisfied. First, if $I_t = 0$ for some $t = \hat{t}$, the Markovian assumption implies search remains zero for $\forall\ t > \hat{t}$. Hence, there are finitely many search periods. Assuming that the game lasts an infinite number of periods, this means that search in some periods will take arbitrarily small amounts, i.e. for $\forall\ \varepsilon_0 > 0\ \exists\ \varepsilon \in (0, \varepsilon_0)$ and $\exists\ t \in N$ such that $I_t = \varepsilon \ll x$. It also means that there will be an infinite number of such periods. From (6) it follows $\Psi(x) = (1 - x)\varepsilon + \delta\Psi(x - J) + o(\varepsilon)$, which can be rewritten as

$$\Psi(x - J) = \frac{\Psi(x) - (1 - x)\varepsilon + o(\varepsilon)}{\delta}, \tag{8}$$

where $J = n\varepsilon + o(\varepsilon)$.

Let us prove that in this case $\Psi(x)$ must be arbitrarily close to zero. Assume on the contrary that $\Psi(x)$ is equal to some positive value that is not infinitely small. By iterating equation (8) a large number of times it is possible to show that $\exists\ \hat{x} \in (0, x)$ such that $\Psi(\hat{x})$ can be larger than any positive value. However, by construction $\Psi(\hat{x}) = \hat{x}V(\hat{x}) \leq \hat{x}$. We have reached a contradiction, proving that

$\Psi(x)$ must be arbitrarily close to zero.

Given $x \leq 7/8$, $\Psi(x)$ can not be arbitrary close to zero, because each player will have incentive to deviate by choosing $I = x - \varepsilon + O(\varepsilon^2)$ and generating $\Psi(x) = (1-x)x + O(\varepsilon) > 0$. Note that this argument works even when x is very small, because a given $x > 0$ can not be infinitely small.

Thus, I_t cannot be arbitrarily close to zero, which means there exists $\zeta > 0$ such that
$I_t > \zeta \ \forall \ t$. Consequently in any non-trivial SMPE the project has to be finished in a finite number of periods. This establishes part (a) of the Lemma.

Given that the project has to be finished in a finite number of periods, backward induction can be applied. Namely, we assume that $\Psi_0 \equiv 0$ and derive $\Psi_1 \equiv B\Psi_0$. This allows us to find all potential SMPE of the game if players could search for at most one period. Then we derive $\Psi_2 \equiv B\Psi_1$, which allows us to find all potential SMPE of the game if players could search for at most two periods. We continue this process until T is reached and find all potential SMPE of the game if players could search for at most T periods. This establishes part (b) of the Lemma and concludes the proof. □

With the help of Lemma 2, one can construct the sequence $\{\Psi_s\}$ and find all SMPE. This value-iteration procedure is equivalent to backward induction.

Proof of Proposition 1

Construction of Ψ_1 and V_1

Let us start from the end of the search process. What will be the value if players could only search for at most one period? Equation (6) transforms into

$$\Psi_1(x) = \max_{I_i \in [0,x]} \{\min\{x, I_i + I_{-i}\} - I_{-i} - xI_i\}. \tag{9}$$

It is evident that in the unique non-trivial SMPE the equilibrium I_i can be described in the following way:

$$I_i = x/n, \ \text{if} \ x \leq 7/8. \tag{10}$$

Given $x \leq 7/8$, in the SMPE players search the whole island, $nI_i = x$. Consequently, the solution of (9) is

$$\Psi_1(x) = P_{(1)}(x), \tag{11}$$

where polynomial $P_{(1)}(x) = x(1-x)/n$. For future reference note that

$$P_{(1)}(x) = \frac{a_1}{n}(1-x)^2 + \frac{b_1}{n}(1-x) + \frac{c_1}{n}, \tag{12}$$

where

$$a_1 = -1, \quad b_1 = 1, \quad c_1 = 0. \tag{13}$$

Define the value for each player (if the players can search the island for at most k periods) as $V_k(x) \equiv \Psi_k(x)/x$, for any $x \geq 0$. From the above definition, it follows that

$$V_1(x) = (1-x)/n.$$

Construction of Ψ_2 and V_2

What will be the value if players can search the island for at most two periods? In general there could be two possibilities, depending on the island size. The first possibility is that the players search the whole island in just one period. Intuitively this happens for small values of x because it is too costly to wait for another period when the island is very small. The second possibility is that the players search the island for at most two periods. This happens for large values of x.

We have already considered the first possibility in the previous section. Now we analyze the situation when players plan to search for at most two periods. The first step is to construct $\Psi_2(x)$. Equation (6) in this case transforms into

$$\Psi_2(x) = \max_{I_i \in [0,x]} \{\min\{x, I_i + I_{-i}\} - I_{-i} - xI_i + \delta\Psi_1\left(\max\{0, x - I_i - I_{-i}\}\right)\}. \tag{14}$$

The necessary condition for I_i to be optimal in the interior of $[0, x]$, provided that search lasts for two periods, is

$$(1-x) - \delta\Psi_1'(x - nI_i) = 0. \tag{15}$$

In order to continue the search for the second period, the remaining island size has to satisfy

$$x - nI_i \geq 0. \tag{16}$$

The sufficient condition for I to be optimal in the interior of $[0, x]$ is satisfied because

$$\Psi_1''(x - nI_i) = a_1 < 0.$$

The way to proceed is to construct the equilibrium with the help of condition (15), and then show that the derived equilibrium satisfies condition (16).

From expressions (15) and (11), it follows that

$$(1 - x) - \delta \left(\frac{1 - 2(x - nI_i)}{n} \right) = 0.$$

Consequently,

$$I_i = \frac{(2\delta - n)x + n - \delta}{2\delta n}. \tag{17}$$

Substituting (17) into equation (14), we obtain a *spline* of degree two on the interval $(0, 7/8]$:

$$\Psi_2(x) = \begin{cases} \Psi_1(x), & \text{if} \quad 0 < x \leq \chi_1, \\ P_{(2)}(x), & \text{if} \quad \chi_1 < x \leq 7/8; \end{cases} \tag{18}$$

where

$$P_{(2)}(x) = \frac{a_2}{n}(1 - x)^2 + \frac{b_2}{n}(1 - x) + \frac{c_2}{n}, \tag{19}$$

with

$$a_2 = -1 - s, \quad b_2 = \frac{1}{2}, \quad c_2 = \frac{\delta}{4}, \tag{20}$$

where

$$s = \frac{n(n-2)}{4\delta}. \tag{21}$$

The point $x = \chi_1$ is the first knot of the spline. When $x \geq \chi_1$ neither player has a unilateral incentive to switch from searching the island for two periods to searching the island for just one period; and $x = \chi_1$ is the point of indifference:

$$\Psi_2(\chi_1) = (1 - \chi_1)(\chi_1 - (n-1)I). \tag{22}$$

On the right hand side of (22), $1-\chi_1$ is the marginal payoff to search in the current period, and $\chi_1 - (n-1)I$ is the additional search intensity required by any one player to complete search immediately. From (17) and (19), we get

$$\chi_1 = 1 - \frac{\delta}{n}. \tag{23}$$

Note that when $x = \chi_1$, one-period search is equivalent to two-period search, as in equilibrium players search nothing in the second period, values in (10) and (17) are the same.

It is straightforward to check that the solution given by (18) satisfies condition (16) for any $x \in [\chi_1, 7/8]$. Therefore, if the players can search the island for at most two periods, the SMPE is a spline of degree one on the interval $(0, 7/8]$ with one possible knot $x = \chi_1$:

$$I(x) = \begin{cases} \frac{x}{n}, & \text{if} \quad x \leq \chi_1, \\ \frac{(2\delta-n)x+n-\delta}{2\delta n}, & \text{if} \quad \chi_1 < x \leq 7/8; \end{cases}$$

and the value function is

$$V_2(x) = \begin{cases} V_1(x), & \text{if} \quad x \leq \chi_1, \\ P_{(2)}(x)/x, & \text{if} \quad \chi_1 < x \leq 7/8. \end{cases}$$

We show below that all search is conducted within two periods and the derived $I(x)$ and $V_2(x)$ are the intensity and the value function of each firm in equilibrium.

Construction of Ψ_3 and V_3

What will be the value if players can search the whole island for at most three periods? In general there could be three possibilities, depending on the island size x_1. The players can plan to search the island for at most $1, 2, 3$ periods in a non-trivial SMPE.

Let us construct $\Psi_3(x)$. Equation (6) in this case transforms into

$$\Psi_3(x) = \max_{I_i \in [0,x]} \{\min\{x, I_i + I_{-i}\} - I_{-i} - xI_i + \delta\Psi_2\left(\max\{0, x - I_i - I_{-i}\}\right)\}. \tag{24}$$

A necessary condition for I_i to be the optimal value in the interior of $[0, x]$, provided that search lasts for up to 3 periods, is

$$(1 - x) = \delta\Psi_2'(x - nI_i). \tag{25}$$

In order to continue search for the next period, the new value of x has to satisfy

$$x - nI_i \geq \chi_1. \tag{26}$$

The sufficient condition for I to be the optimal value in the interior of $[0, x]$ is satisfied if

$$\Psi_2''(x - nI_i) < 0. \tag{27}$$

We will use condition (25) to find I, and then show that it satisfies conditions (26) and (27). Note that if function $\Psi_2(x)$ in (24) is a quadratic polynomial, then $\Psi_3(x) = B\Psi_2(x)$ has to be a quadratic polynomial as well. Since $P_{(1)}(x)$ and $P_{(2)}(x)$ are quadratic polynomials by (12) and (19), $P_{(3)}(x)$ can be represented in the following form:

$$P_{(3)}(x) = \frac{a_3}{n}(1 - x)^2 + \frac{b_3}{n}(1 - x) + \frac{c_3}{n}. \tag{28}$$

Let $I_{(k)}$ be the optimal search intensity of a rival who anticipates search for up to k periods. From condition (25) and expression (28), it follows that

$$I_{(3)} = -\frac{(1 - x)(n + 2\delta a_2)}{2\delta n a_2} - \frac{b_2}{2na_2}. \tag{29}$$

At the knot $x = \chi_2$, each player is indifferent between planning to search the area for up to 3 periods and unilaterally accelerating search to ensure that it is concluded within 2 periods:[14]

$$\Psi_3(\chi_2) = \Psi_2(\chi_2) + n(1 - \chi_2)(I_{(2)} - I_{(3)}). \tag{30}$$

[14] An alternative deviation is to intensify search more dramatically so that search is completed within 1 period. We showed earlier that accelerating search to ensure that it is concluded within 1 period is dominated by searching within 2 periods.

On the right hand side of (30), $1-\chi_2$ is the marginal payoff to search in the current period and $n(I_{(2)} - I_{(3)})$ is the additional search intensity required in the current period to unilaterally shift to an accelerated search plan that will be concluded within 2 periods.

Equation (30) has two solution. The smaller value of χ_2 corresponds to searching negative amount in the first period and so it does not correspond to any solution. Note also that when $x = \chi_2$, there are two equilibria corresponding to planning to search the area for 3 periods or for 2 periods as the condition (30) is the same condition that describes potential deviations from both equilibria.

Substituting (29) into equation (24), we obtain a *spline* of degree two on the interval $(0, 7/8]$ with knots χ_1 and χ_2:

$$\Psi_3(x) = \begin{cases} \Psi_2(x), & \text{if} \quad 0 < x \leq \chi_2, \\ P_{(3)}(x), & \text{if} \quad \chi_2 < x \leq 7/8; \end{cases} \tag{31}$$

where $P_{(3)}(x)$ is defined in (28). Therefore, if players plan to search the island for at most 3 periods, then $I(x)$ is a spline of degree one on the interval $(0, 7/8]$ with knots χ_1 and χ_2:

$$I(x) = \begin{cases} \frac{x}{n}, & \text{if} \quad x \leq \chi_1, \\ -\frac{(1-x)(n+2\delta a_1)}{2n\delta a_1} - \frac{b_1}{2na_1}, & \text{if} \quad \chi_1 < x \leq \chi_2, \\ -\frac{(1-x)(n+2\delta a_2)}{2n\delta a_2} - \frac{b_2}{2na_2}, & \text{if} \quad \chi_2 < x \leq 7/8; \end{cases}$$

and the value function is

$$V_3(x) = \begin{cases} V_2(x), & \text{if} \quad 0 < x \leq \chi_2, \\ P_{(3)}(x)/x, & \text{if} \quad \chi_2 < x \leq 1. \end{cases}$$

Let us now find a_3, b_3, and c_3. Using (24), (28) and (29), we get the following values:

$$a_3 = -\frac{1+2s}{1+s}, \quad b_3 = \frac{1}{4(1+s)}, \quad c_3 = \frac{\delta^2}{4} + \frac{\delta}{16(1+s)}. \tag{32}$$

Derivation of χ_2 and the maximum search time

To find χ_2, one needs to solve the quadratic equation $P_{(3)}(\chi_2) = P_{(2)}(\chi_2) + n(1 - \chi_2)(I_{(2)} - I_{(3)})$, namely

$$a_3(1-\chi_2)^2 + b_3(1-\chi_2) + c_3 = a_2(1-\chi_2)^2 + b_2(1-\chi_2) + c_2 + n(1-\chi_2)(I_{(2)} - I_{(3)}).$$

Simplifying gives

$$(n-4)(a_3 - a_2)(1-\chi_2)^2/(n-2) + 2(b_3 - b_2)(1-\chi_2) + (c_3 - c_2) = 0. \tag{33}$$

When $n = 2$ and n=4 the quadratic term cancels out; consequently,

$$\chi_2 = 1 + \frac{c_3 - c_2}{2(b_3 - b_2)}. \tag{34}$$

Substitute values from (20) and (32) to simplify

$$\chi_2 = 1 - \frac{\delta - 4\delta(1-\delta)(1+s)}{8(1+2s)}. \tag{35}$$

As $\delta < 1$ and $s \geq 0$, one can directly show that $\chi_2 > 7/8$. To extend this result for $n = 3$, note that in (33) the first term is always negative as $a_3 - a_2 = \frac{s^2}{1+s} > 0$, while the second terms is negative as $b_3 - b_2 < 0$. Note also that we showed earlier that $\chi_2 > 7/8$ when the first term in (33) is zero. Adding a negative term can only increase χ_2.

Now let us consider when $n > 4$. Substitute values from (20) and (32) to derive

$$n^2(n-2)(n-4)(1-\chi_2)^2 - 4\delta(2\delta + n(n-2))(1-\chi_2) + \delta^2(\delta-1)(4\delta + n(n-2)) + \delta^3 = 0. \tag{36}$$

There are two roots to the above equation. As discussed earlier, the smaller root is consistent with searching negative amount in the first period. Consider the bigger root. Notice that when $n \to \infty$, $\chi_2 \to 1$. On the other hand, $\chi_2 \geq 7/8$ when $n \leq 4$. Notice also that χ_2 is implicitly defined by (36) and is a continuous function of n. Substitution of $\chi_2 = 7/8$, results in equation in terms of n, which can be simplified to

$$n(n-4)/64 - \delta/2 - \delta^2 + \delta^3 = \delta^2(1-\delta)(4\delta+1)/(n(n-2)). \tag{37}$$

The left hand side of (37) is increasing in n, while the right hand side is decreasing in n. There is a unique value of n that solves (37). As there are two roots to (36), this solution corresponds to the smaller root, which does not satisfy the constraint that search has to be positive. Consequently, $\chi_2 > 7/8$ for any n. This means that all search is conducted within two periods.

Constraints

Define that part of x which player i does not search in the current period by $y = x - I_i$ and the part of x no player searches in the current period by $z = x - (I_i + I_{-i})$.

First, let us prove that for any value of x, (31) satisfies condition(27); that is, $\Psi_2''(z) < 0$. From equation (31), it is clear that the sufficient condition for $\Psi_2''(z) < 0$ is that $P_{(i-1)}''(z) < 0$ for $i = 2$ and $i = 3$. From equation (28), it is easy to see that the above condition is equivalent to $a_{i-1} < 0$ for $i = 2$ and $i = 3$, which is satisfied, see (13) and (20).

Second, note that proving the condition (26) in unnecessary as for values $x \leq 7/8$ three period search is dominated by searching in two periods. This final observation concludes the proof. □

Proof of Proposition 2

In the proof below we follow the argument presented in the proof of Proposition 1 in Ericson and Pakes (1995).[15] Introduce the following operator $\tilde{B}$ defined by

$$(\tilde{B}V)(x) \equiv \max_{I_i \in [0,x]} \left\{ -I_i + \frac{\hat{J} - \hat{I}_{-i}}{x} + \delta \left(1 - \frac{\hat{J}}{x} \right) V(x - I_i) \right\}, \tag{38}$$

where the solution has to satisfy $I_i = I$ and $\hat{J} = I_i(1 - I/x)^{n-1} + \hat{I}_{-i}$. By the contraction property of the operator defined in equation (38), $V(x) = \lim_{s \to \infty} V_s(x)$, where $V_s(x) \equiv \tilde{B}V_{s-1}(x) \equiv \tilde{B}^s V_0$ and $V_0 \equiv 0$. The remainder of the proof follows

[15] Also see Stokey, Lucas and Prescott (1989).

by induction. $V_0 \equiv 0$, so an induction argument for $s = 0$ is satisfied. Assume that $V_s \equiv 0$, and let us show that $V_{s+1} \equiv 0$. Substituting $V_s \equiv 0$ into the right hand side of equation (38), cancels out the continuation term on the right, yielding

$$V_{s+1}(x) = \tilde{B}V_s(x) \equiv \max_{I_i \in [0,x]} \left\{ -I_i + \frac{I_i}{x} \left(1 - \frac{I}{x} \right)^{n-1} \right\}. \quad (39)$$

First, let us prove that $x - x^{n/(n-1)} < 7/8$. As $x \leq 7/8$, it follows that $x^2 < x^{n/(n-1)}$. Consequently, it is sufficient to prove that $x - x^2 < 7/8$, which trivially holds.

Next, if the search by each other player $I < x - x^{n/(n-1)}$, it is a dominant strategy for player i to choose $I_i(x) = x$. If, on the other hand, search by each other player $I > x - x^{n/(n-1)}$, it is a dominant strategy for player i to opt for $I_i(x) = 0$. Moreover, if search by each other player $I = x - x^{n/(n-1)}$, then player i is indifferent among all $I_i \in [0, x]$. There is a unique symmetric equilibrium where each player searches $I(x) = x - x^{n/(n-1)}$ and the value each receives is zero; that is, $V_{s+1} \equiv 0$. Taking the limit gives $V(x) = \lim_{s\to\infty} V_s(x) \equiv 0$. This concludes the proof. □

Proof of Lemma 1

Suppose that the statement of the lemma is not satisfied and there is an equilibrium with $V(x_1) > V^{(n)}(x_1)$. Firstly, notice that in this equilibrium there should be no duplication. As players are getting a zero value when they duplicate, they have an incentive to search somewhere else when they can.

Secondly, consider a path without duplication that delivers $V(x_1) > V^{(n)}(x_1)$. The same path will be an equilibrium in the game when search is observable, leading to a contradiction. □

Proof of Proposition 3

First, in the proposed equilibrium all players search $I^{(n)}(x)$, which was derived in Proposition 1. Specifically,

$$I^{(n)}(x) = \begin{cases} \frac{x}{n}, & \text{if} \quad x \leq 1 - \frac{\delta}{n}, \\ \frac{(1-x)(n-2\delta)+\delta}{2\delta n}, & \text{if} \quad 1 - \frac{\delta}{n} < x \leq \frac{7}{8}. \end{cases} \tag{40}$$

Players have no incentive to deviate by changing the division of search between the first and later periods as strategy $I^{(n)}(x)$ is an equilibrium search strategy for the game when actions are observable. Second, let us also show that neither player has an incentive to search the area associated with the other players. Searching this area in the first period by ε would generate an additional payoff of

$$-\varepsilon + \frac{\varepsilon(x - nI^{(n)})}{x^2}. \tag{41}$$

The first term is the cost of searching the other players' area, while the second term is the expected benefit to the deviator who searches ε, while the 'incumbent' searches $I^{(n)}$ in his area x/n. It is assumed that from the deviator's prospective, the 'incumbent' searches his area randomly. The combined effect given by (41) is non-positive for any positive ε when

$$I^{(n)} \geq (x - x^2)/n. \tag{42}$$

Let us show that (42) always holds. Note that $I^{(n)}(x)$ given by (40) is weakly decreasing in δ and increasing in n for any value of x. This means that (42) is hardest to satisfy when $\delta = 1$ and $n = 2$. Substituting $\delta = 1$ and $n = 2$ into (40) and applying (42) transforms to the following inequalities

$$\begin{cases} \frac{x}{n} \geq \frac{x-x^2}{n}, & \text{if} \quad x \leq 1 - \frac{\delta}{n}, \\ \frac{1}{2n} \geq \frac{x-x^2}{n}, & \text{if} \quad 1 - \frac{\delta}{n} < x \leq \frac{7}{8}, \end{cases} \tag{43}$$

which always hold. This concludes the proof. □

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