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Collusion with intertemporal price dispersion

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Abstract

We develop a theory of optimal collusive intertemporal price dispersion. Dispersion clouds consumer price awareness, encouraging firms to coordinate on dispersed prices. Our theory generates a collusive rationale for price cycles and sales. Patient firms can support optimal collusion at the monopoly price. For less patient firms, monopoly prices must be punctuated with fleeting sales. The most robust structure involves asymmetric price cycles resembling Edgeworth cycles. Low consumer attentiveness enhances the effectiveness of price dispersion by reducing the payoff to deviations involving price reductions. However, for sufficiently low attentiveness, price rises are also a concern, limiting the power of obfuscation.

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The nagging presence of competitors is an inescapable fact of life for most firms. To understand pricing behaviour, we must therefore consider the implications of repeated interaction. The theory of repeated games offers some insights; notably, repeated play opens up opportunities for punishments and rewards, permitting higher prices to be supported. However, non-trivial price dynamics are also a common outcome of repeated play. For example, using scanner data covering 1.4 million goods in 54 geographic markets, Kaplan and Menzio (2015) find that close to half the variation in prices is intertemporal.[1] An even larger role is typically played by intertemporal price variation in retail gasoline markets subject to regular price cycles known as Edgeworth cycles.[2]

The rationale for these pricing patterns is not obvious in a repeated game setting. If the threat of punishment is available, why don't firms employ it to support the highest feasible fixed price? Complicated pricing patterns may be both more difficult to coordinate on and less profitable. In this paper we develop a theory of coordinated price dispersion that provides an intuitive explanation. Our theory rationalises commonly observed pricing patterns including sales, price cycles, and fixed prices. Our starting point is the consumer.

Pretend for a moment that you are a consumer. Think of a few products that may be in your shopping basket; for example, milk, coffee, petrol, breakfast cereals. What is the current price of these items at your local store? What is the current price at other stores? Would you recognise a bargain? Your answers to these questions may vary by product; frequency of purchase and prominence of display are obvious factors. We conjecture that the complexity of pricing patterns also plays a role. A consumer who routinely observes a single price may become accustomed to that price and recognise immediately a departure from this simple pricing pattern. By contrast, complicated price paths are more difficult to absorb, making price changes less obvious.[3] If consumer price perceptions are influenced by pricing patterns in this manner, they may be ripe for manipulation by firms. We consider the optimal

[1] Kaplan and Menzio (2015) use the Kilts-Nielsen Consumer Panel Dataset, which records the shopping behaviour of approximately 50,000 consumers over the period 2004-2009, and contains over 300 million transactions. The authors decompose the variation in prices into that due to variation across stores, across products at a given store, and across transactions over time for a specific product-store pair, finding that intertemporal variation in prices accounts for close to half the variation.

[2] For example, in the Perth gasoline market studied by Wang (2009) and de Roos and Katayama (2013), a similar decomposition reveals that in 2003, approximately 54% of the variation in retail margins is accounted for by intertemporal variation in margins at a given retail station. If we condition only on stations participating in the cycle (the majority of stations), this rises to around 97%. Further details are available on request.

[3] This line of reasoning is supported by the concepts of rehearsal and associativeness that Mullainathan (2002) draws from the literature on memory research. Taken together, these two principles may imply a constant price would become a well established reference for comparison, while an intertemporally dispersed price path might be less amenable to these memory processes.

pricing problem of a cartel faced with this prospect.

In our model, each consumer accumulates pricing information about a specific product both when she is shopping for the product and when she is not. While she is not actively engaged in search, information gathering is a background process subject to limited memory. We incorporate bounded recall and finite memory in a manner reminiscent of the sampling process used in a substantial literature on adaptive play (Young, 1993). Each consumer recalls only a subset of the current price vector, with prices that are unusually low more susceptible to recall. In our base specification, she recalls all prices below a threshold y, while prices above y are recalled with probability less than one. The threshold is based on the lowest price observed by consumers in the previous m periods. When the need to purchase becomes her focus, she engages in a process of costly sequential search with this partial price information as a starting point.

Firms compete in the market for a homogeneous product by simultaneously setting price over an infinite horizon. The immediate problem for any cartel is to maintain incentives for internal discipline. In our model, the price path chosen by the cartel determines the prices experienced by consumers and thus the incentive to deviate. If prices are intertemporally dispersed and consumers are accustomed to seeing prices over a wide interval, then most prices will be above y and demand may not be responsive to price differences. By contrast, if prices are fixed over time, even modest price cuts might trigger consumer attention. Price dispersion then plays a useful role for the cartel as an obfuscation device. With dispersion, consumers are less responsive to price changes, the payoff to deviating from cartel policies is reduced, and the cartel's internal incentive constraints are relaxed. The optimal dynamic price path for the cartel then emerges as a trade off between profitability and obfuscation.

In addition to cartel manipulation, innate market characteristics influence a consumer's tendency to perceive and recall price information. We parameterise our model by the level of consumer attentiveness, ranging from perfect awareness to complete inattentiveness. At one extreme, our specification approaches the Bertrand model and obfuscation is futile. Consumers are aware of all prices and an undercutting firm captures the whole market independent of the extent of intertemporal price dispersion. At the other extreme, if consumers are not aware of any specific prices when search begins, our model replicates the Diamond (1971) paradox: if all firms were to set a price below the monopoly price in a single-period version of our model, then there would be an incentive to raise price.

Let us briefly preview some of our main findings. Our most fundamental result is that symmetric collusive equilibria exist for a greater range of discount factors with an intertemporally dispersed price path than with a fixed price. The optimal price path reflects a compromise between profitability and the obfuscation properties of the path. Every sequence begins with the monopoly price and then follows a weakly decreasing trajectory; each subsequent period involves either monopoly pricing or a strict price reduction. If cartel members are sufficiently patient, setting a fixed price at the monopoly level is optimal. However, for lower values of the discount factor, intertemporally dispersed prices are required to dampen the market share benefits for potential deviators.

Two specific equilibrium outcomes of our model are closely related to commonly observed pricing dynamics. First, for a range of parameter values, we observe sales: prolonged periods of monopoly pricing interspersed with large temporary price reductions. One-period sales present the cheapest means of delivering price dispersion. If firms are less patient, the length and depth of sales must increase in order to satisfy the incentive constraints of the cartel. Second, sales of maximum length and depth afford the greatest protection against deviation. These pricing paths resemble asymmetric price cycles known as Edgeworth cycles.

In markets with alert consumers the benefits of price dispersion for cartel sustainability are minimal. As attentiveness is reduced, obfuscation becomes more effective and cartel sustainability is improved. However, there is a limit to this process. For low levels of attentiveness and high search costs, the primary threat to cartel stability is not the temptation to undercut, but rather the incentive to raise prices above the levels dictated by the cartel. If consumers pay little heed to prices, then the penalty for doing so may be minimal. Consequently, if attentiveness levels are sufficiently low, the relationship between cartel sustainability and attentiveness may be reversed: as attentiveness is reduced, upward price deviations become more attractive and the cartel becomes *harder* to sustain.

The idea that consumers are imperfectly informed about prices is not new. For example, a substantial literature on consumer search demonstrates that imperfect information can have dramatic consequences for pricing.[4] If consumers are imperfectly informed, this disturbs a fundamental pricing trade-off faced by firms: prices balance the lure of extracting rents from consumers with the incentive to wrest consumers from competitors and defend against rival behaviour. If consumers are poorly informed, the balance may be tilted towards rent extraction and higher prices. Our departure is to allow a consumer's price awareness to be coloured by her experience with prices. Unusual or attractive prices, *relative to experi-*

[4] See Baye et al. (2006) for a recent survey. Note that the interplay between collusion and price dispersion in our model is richer than in a typical consumer search setting. Contemporaneous price dispersion is potentially harmful for a cartel because it raises the benefits of search, improving consumer price awareness and making the cartel's incentive constraints more difficult to satisfy. This mechanism also applies in our theory. In addition, intertemporal price dispersion reduces consumer price awareness, relaxing the cartel's incentive constraints.

ence, are more likely to trigger consumer attention. Firms can then collectively adjust the pricing trade-off by influencing the history of prices consumers experience.

In the theory of rational inattention (Sims, 1998, 2003), consumers have limited information processing capacity, and optimally deploy this capacity.[5] For example, in the discrete choice problem of Matějka and McKay (2015), consumers decide how to sharpen their prior beliefs about the value of n alternatives based on the costs of obtaining information. The information acquisition strategy is specific to the current decision problem. Our model offers an alternative interpretation in which the attention process stems from the accumulated experience of consumers over time and across markets. Nevertheless, the advantage of our simple threshold rule in the immediate problem is clear. Unusually low prices are of the greatest benefit in costly sequential search, and it may be advantageous to apportion costly memory resources to these prices. In extensions to our main model (Section 4.1), we show that our conclusions are not sensitive to our specification of the attention process.

Our work has antecedents in Range Theory (Volkmann, 1951), which suggests that the attractiveness of a specific market price is relative to the lower and upper bound of price expectations consumers form through experience.[6] Similarly, consumers in our model are more likely to retain the price information of a specific vendor if the offered price is unusual relative to their experience.

We contribute to a growing literature on obfuscation in oligopoly. Obfuscation operates by hampering consumer efforts to directly compare competing products. For example, obfuscation could take the form of additional noise in the price process (Spiegler, 2006), the choice of price formats with different comparability properties (Piccione and Spiegler, 2012; Chioveanu and Zhou, 2013), the shrouding or emphasis of product attributes (Gabaix and Laibson, 2006; Bordalo et al., 2013), or attempts to raise the cost of consumer search (Ellison and Wolitzky, 2012; Wilson, 2010).[7] Our principal conceptual departure from this literature

[5]Recent examples suggest inattention can be extreme. Clerides and Courty (2017) document that some consumers do not make basic price comparisons even when the information is right before them. Monroe and Lee (1999) argue that consumers often do not remember specific prices for products they have purchased, but they nevertheless have an intuitive understanding about prices. Mazar et al. (2014) present experimental evidence suggesting revealed preferences depend on price experience unless subjects are explicitly prompted to evaluate their decision making.

[6]Range Theory has some recent observational support. For example, Janiszewski and Lichtenstein (1999) present experimental evidence that variation in the width of the price range affects price-attractiveness judgments, independent of changes to a consumer's reference price. Relatedly, Moon and Voss (2009) find that a model based on range theory provides additional explanatory power in the presence of reference price theories in a panel data setting.

[7]Our search mechanism is most closely related to the setting considered by Spiegler (2006), in which consumers adopt a simple search heuristic: they sample the price of each firm, identify the cheapest offering, and

is to offer a motivation for *intertemporal* obfuscation. Coordination on price may be insufficient to satisfy the incentive constraints of a cartel. We show that coordination on obfuscation may relax these incentive constraints. In our theory, intertemporal price variation acts as the obfuscation mechanism; we expect alternative mechanisms to yield similar insights.

The problem of a cartel with opportunities to exploit imperfections in the consumer choice process is also explored by de Roos (2018) and de Roos and Smirnov (2015). In de Roos and Smirnov (2015), we extend our current theme by considering a cartel with incomplete membership. Faced with a fringe competitor, the cartel has an additional motivation for obfuscation: to limit the ability of the fringe to undercut the cartel price and steal market share. de Roos (2018) adopts a different approach. Cartel members choose the manner in which their product is presented or framed to consumers, lending the cartel the collective ability to manipulate the propensity of consumers to compare competing products. Obfuscation aids the cartel by reducing the effectiveness of deviation, but may also constrain the severity of punishment. de Roos (2018) analyses the resulting trade-off between deviation and punishment. By contrast, in the current article, consumer choice is impacted by the history of prices, and we examine the cartel's optimal dynamic pricing problem.

We are not the first to examine cartel pricing dynamics. However, most explanations for cartel pricing dynamics involve exogenous market processes; examples include demand side dynamics (Green and Porter, 1984; Rotemberg and Saloner, 1986), and entry, exit, and investment dynamics (Fershtman and Pakes, 2000; de Roos, 2004). Nava and Schiraldi (2014) provide a recent exception. The authors show that sales are optimal for a cartel if consumers can store for future consumption.[8] In this setting, sales reduce the incentive to deviate from cartel prices in both non-sale periods (if some consumers have storage constraints) and sale periods (because prices are lower and consumers anticipate a price war following deviation).[9] In our model, the desire for obfuscation rather than consumer storage gives rise to price dynamics, and sales and Edgeworth cycles emerge as special cases.[10]

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then return to buy from this firm. Unfortunately (for consumers), firms adopt dispersed prices in equilibrium, and the firm consumers return to may no longer offer the cheapest price.

[8]Another example of cartel price dynamics in a stationary environment is the cartel's optimal pricing problem in the presence of an anti-trust authority (Harrington, 2004, 2005). In that case, price dynamics are temporary as the cartel transitions towards optimal collusive pricing under the watchful eyes of the anti-trust authority. By contrast, we observe lasting intertemporal price variation.

[9]In Section 4.3, we allow a measure of consumers to time their purchases; equivalently, they have access to free storage. This impacts the optimal path by accentuating the depth of sales, but does not influence the sustainability of collusion.

[10]In our theory, firms are intertemporal optimisers and consumers behave myopically. Fershtman and Fishman (1992) demonstrate that non-trivial price dynamics can arise when these roles are reversed.

Sales are also associated with price discrimination.[11] For example, Sobel (1984) suggests sales as a method to discriminate between customers with high and low reservation prices, while in Varian (1980), sales discriminate between informed and uninformed consumers. We suggest obfuscation as an alternative motivation.[12] A brief sale can generate a wide interval of prices, hampering consumer efforts at price comparison at minimal cost to the cartel.

We also provide an explanation for price cycles. Asymmetric cycles that resemble Edgeworth cycles provide the greatest internal stability for the cartel. Edgeworth cycles are a striking feature of retail petrol markets in a number of countries including Canada (Noel, 2007b; Eckert and West, 2004), the United States (Lewis, 2012), Australia (Wang, 2009), and Norway (Foros and Steen, 2008). The most commonly cited explanation for Edgeworth cycles is the price commitment model of Maskin and Tirole (1988).[13] Asynchronous price setting and the restriction to Markov strategies are central to the predictions of their model, restrictions that are absent in our model. However, coordination and simultaneous play are attracting growing empirical support in retail petrol markets subject to cyclical pricing. First, the limited evidence on the timing of play suggests simultaneous price setting (de Roos and Katayama, 2013). Second, evidence of collusion has emerged. Explicit communication is evident in Ballarat, Australia (Wang, 2008) and Québec (Clark and Houde, 2013), while Byrne and de Roos (2015) provide evidence of tacit collusion in Perth, Australia.

The rest of the paper is structured as follows. In Section 1, we introduce the model and describe the implications of consumer inattention for demand. We begin the analysis in Section 2 with a restricted problem in which the cartel seeks the optimal infinitely repeated cyclical pricing strategies. Our main results for the paper are contained in Lemma 4, where we provide a detailed characterisation of the optimal price path. In Section 3, we confirm that cyclical strategies are optimal when consumer memory is finite. The discussion in Section 4 examines robustness to alternative specifications of consumer attentiveness, asymmetric and mixed strategies, and forward-looking consumers. All proofs are in Appendix A.

[11] Albrecht et al. (2013) also highlight the role of search frictions in generating sales. Other explanations include demand anticipation (Salop and Stiglitz, 1982), introductory offers for experience goods (Doyle, 1986), loss aversion (Heidhues and Kőszegi, 2014), and manipulation of consumer reference prices (Bordalo et al., 2013).

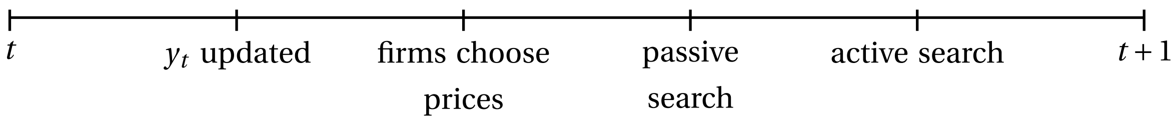
[12] While there is no role for price discrimination in our model, obfuscation and price discrimination motives for sales could be complementary.

[13] A class of alternative explanations for the cycle includes the original Edgeworth (1925) model of capacity constrained price competition. The principle components are a discontinuity in residual demand and positive residual demand above the discontinuity. A potential limitation of this class of explanation is that, to explain a cycle as an equilibrium phenomenon, we must assume that firms play a myopic best response to their rivals' previous prices. See de Roos (2012) for details.

1 The model

A market comprises a set of n firms, indexed by $j = 1, \ldots, n$, selling an undifferentiated product to a continuum of consumers with unit mass, indexed by $i \in [0,1]$. In each period $t = 0, \ldots, \infty$, firms simultaneously choose prices, leading to a price vector $\mathbf{p} = (p^1, \ldots, p^n)$. Each consumer then chooses between the n offerings. Figure 1 summarises the sequence of events taking place each period, to be detailed below. In the following sections, we discuss the consumer decision-making process, the dynamic problem faced by firms, and the definition of equilibrium.

Figure 1: Sequence of events in each period



1.1 Consumers

Consumers are not strategic players; they myopically maximise surplus in the current period. Each consumer is willing to buy a single unit of the good up to a choke price of 1.[14] She makes her selection according to a two-stage search process.

First, while potentially engaged in other activities, she accumulates background information about prices in a stage we label *passive search*. Due to direct observation, word of mouth, or advertising, she is exposed to the current price vector.[15] Because product selection is not her current focus, she does not retain all of this information. Her propensity to recall each price depends on the salience of the price given her recent price experience. In our base specification, the probability of recalling price p is determined by the salience relation $\phi(p, y)$ given by

$$\phi(p,y) = \begin{cases} \beta & \text{if } p > y; \\ 1 & \text{if } p \le y, \end{cases} \tag{1}$$

[14]The assumption of unit demand is not required. Equivalently, we could specify download-sloping consumer-level demand with a unique monopoly price.

[15]In our main specification, consumers are exposed to all prices. This assumption simplifies the analysis, but is not required for our main conclusions. The relevant detail for analysis is consumer exposure to the prices of deviating firms, and we might expect undercutting firms to make a greater effort to expose consumers to their prices. For example, firms may advertise their prices when undercutting. Example 5 in Section 4.1 is consistent with partial price exposure.

where $\beta \in (0,1)$ is a constant and the period t cut-off price y_t depends on prices in the previous m periods according to

$$y_t = \min_{\tau \in \{t-m,\ldots,t-1\},j} p_\tau^j. \tag{2}$$

Thus, her attention gravitates towards unusually low prices.[16] The parameter β captures the latent attentiveness of consumers towards prices while engaged in other activities. The parameter m determines the limits to consumer memory.

Following passive search, *active search* begins, involving costly sequential search with free recall. Initially, a consumer can either purchase from any of the stores with salient prices (if there were any), or search for another store at a cost of $c > 0$. In each subsequent round, she can either accept the price of a previously visited or salient store, or continue searching at a further cost of c. Ties are resolved by randomisation. This two stage search process matches that of Robert and Stahl (1993), except that the initial price information retained by consumers is determined by salience rather than the intensity of advertising.[17]

Consumer search decisions are based on a comparison of search costs with expected search benefits. Let F_t be the period t distribution of prices in equilibrium, and suppose that through passive and active search, p is the lowest price recalled by consumer i up to the current round of search (with $p = \infty$ if no prices are recalled).[18] Her expected benefit from searching and then buying from the lowest price available is

$$W_t(p) = \int_0^p \left(p - x\right)\, dF_t(x),$$

Her indifference point defines her reservation price as a function of search costs, $z_t(c)$:

$$W_t(z_t) = \int_0^{z_t} (z_t - x)\, dF_t(x) = c. \tag{3}$$

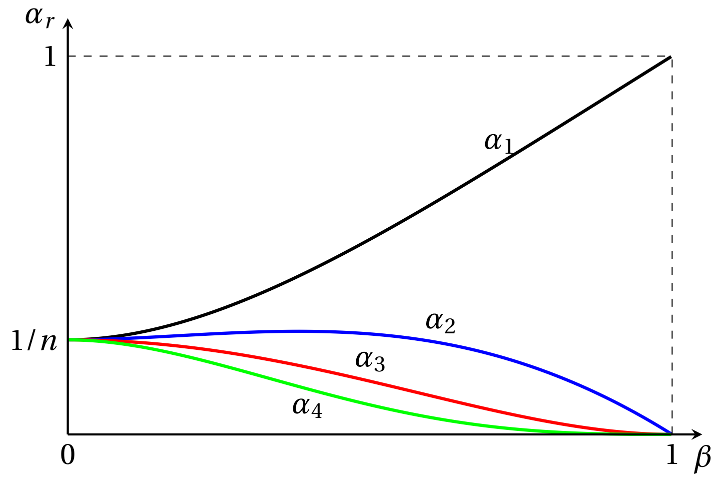
Consumer choices are then summarised by the sequence of reservation prices $\{z_t\}_{t=0}^{\infty}$. It is well known that z_t is independent of the number of remaining unsearched stores (see, for

[16] It is worth emphasising that our results are not overly sensitive to this specification of consumer attentiveness. In early versions of the paper, consumer attention was piqued by unusually low *and* high prices, with no qualitative impact on model predictions. In Section 4.1, we consider alternative generalisations of consumer attentiveness.

[17] Haan and Moraga-Gonzàlez (2011) also consider the relationship between salience and search. Consumers are more likely to begin search at firms with advertising that is salient to them. In their setting, salience is determined by the intensity of advertising, while salience is determined by price in ours.

[18] In a symmetric equilibrium, consumers sample from the same distribution F_t in each round of search in period t. When we consider asymmetric equilibria in Section 4.2, we shut down active search by imposing sufficiently high search costs.

Figure 2: Market shares by attentiveness, $n = 4$



example, Stahl (1989) and Kohn and Shavell (1974)).[19]

We now consider the implications of consumer search for demand. Let $p_{(r)}$ be the r^{th} lowest price in $\mathbf{p}$. It follows directly from (1) that, if firm j uniquely sets price $p_{(1)}$ and $p_{(1)} \leq y$, then she obtains a market share of 1. When instead $p_{(1)} > y$, we use the notation α_r to denote the market share of a firm uniquely setting price $p_{(r)}$, for $r = 1, \ldots, n$. If all firms price below the reservation price z, then

$$\alpha_r = (1-\beta)^{r-1}\beta + (1-\beta)^n / n. \tag{4}$$

A consumer will purchase at a price $p_{(r)}$ if the $r-1$ lower prices are not salient and either $p_{(r)}$ is salient or none of the prices are salient. In the latter case, the consumer engages in additional rounds of active search, selecting with equal probability each store setting a price no higher than z.

Figure 2 illustrates market shares for a range of attentiveness parameters, β. For markets with a high degree of attentiveness, there is a greater market share payoff to undercutting the prices of one's rivals (α_1 is higher), and a smaller payoff to being a high-priced outlier (α_n is lower). When $\beta = 1$, our model collapses to a reduced form of the perfect information case with $\alpha_1 = 1$ and $\alpha_r = 0$, $r = 2, \ldots, n$. When $\beta = 0$, consumers are completely inattentive towards prices above the threshold, and $\alpha_r = 1/n$ for all r.

Lemma 1 summarises the discussion. As is standard in repeated games, we consider unilateral deviations; and as is standard in consumer search, firms have no incentive to price above the reservation price. The market shares α_r in (4) are therefore sufficient for analy-

[19]Alternatively, if search is without recall, reservation prices (i) are higher than under the free recall case and (ii) increase as the number of unsearched firms dwindles, with no qualitative implications for our results.

sis. For completeness, in the lemma we resolve ties uniformly and detail the implications of pricing above the reservation price.

Lemma 1. *Given* $\mathbf{p} \in [0,1]^n$*, reservation price* z*, and cut-off price* y*, the market share of firm* j *is given by*

$$s^j(\boldsymbol{p}, y, z) = \begin{cases} 0 & \text{if } p^j > z \text{ or } \left(p_{(1)} \le y \text{ and } p^j > p_{(1)}\right); \\ \left(\sum_{r=q}^{q+l-1} \alpha_r\right)/l & \text{if} \begin{cases} p_{(1)} > y \text{ and } p^j = p_{(q)} = p_{(q+l-1)} \text{ and} \\ \left(q = 1 \text{ or } p_{(q-1)} < p_{(q)}\right) \text{ and} \\ \left(q+l-1 = n \text{ or } p_{(q)} < p_{(q+l)}\right); \end{cases} \\ 1/l & \text{if } p^j = p_{(1)} = p_{(l)} \le y \text{ and } \left(l = n \text{ or } p_{(l)} < p_{(l+1)}\right), \end{cases} \tag{5}$$

where, for $r = 1, \ldots, n$*,*

$$\alpha_r = \begin{cases} (1-\beta)^{r-1}\beta + (1-\beta)^n/n & \text{if } p_{(n)} \le z; \\ (1-\beta)^{r-1}\beta + (1-\beta)^q/q & \text{if } p_{(q)} \le z < p_{(q+1)},\ r \le q < n. \end{cases} \tag{6}$$

Remark 1. All remaining analysis holds for any market shares $\{\alpha_r\}_{r=1}^n$ such that: i) $\alpha_r > \alpha_{r+1}$, $r = 1, \ldots, n-1$ and $\sum_{r=1}^n \alpha_r = 1$; ii) $\alpha_n(\beta) \in [0, 1/n]$ and $\alpha_1(\beta) \in [1/n, 1]$; iii) $\alpha_n(0) = \alpha_1(0) = 1/n$; iv) $\alpha_n(1) = 0$ and $\alpha_1(1) = 1$; and v) $\alpha_n'(\beta) < 0$ and $\alpha_1'(\beta) > 0$.

Lemma 1 implies that we can focus on the market share function described in (4) as the elementary description of consumer behaviour. Remark 1 clarifies that our analysis does not rely on the model of consumer search, but also encompasses alternative processes for market shares that satisfy the above properties. For symmetric cartels, the viability of collusion depends only on the market share of a firm undercutting the cartel price, α_1, and the market share of a firm deviating by raising price, α_n. Thus, an alternative theory of consumer behaviour need only specify α_1 and α_n. For example, a minor perturbation to the Bertrand rationing rule in which the lowest priced firm does not capture the entire market would be consistent with our analysis.

1.2 Firms

Firms discount the future at the common rate $\delta \in (0,1)$ and have constant marginal costs which we normalise to zero. Profits for firm j are given by

$$\pi^j(\mathbf{p}, y, z) = p^j s^j(\mathbf{p}, y, z). \tag{7}$$

Each firm j simultaneously chooses a price $p^j \in A_j = \Re_+$, with set of pure action profiles $A = \prod_j A_j$. A pure strategy for firm j is a mapping from the set of all possible histories to the set of actions. Let $\mathscr{H}^t = A^t$ be the set of t-period histories, and $\mathscr{H}^0 = \{\emptyset\}$ define the initial history. The set of possible histories is then $\mathscr{H} = \bigcup_{t=0}^{\infty} \mathscr{H}^t$, and a strategy for firm j is a mapping $\sigma^j : \mathscr{H} \to A_j$.[20] Given the strategy profile σ, let $a_t^j(\sigma)$ be the period t action for firm j induced by σ, and let $a_t(\sigma)$ be the associated action profile. We denote by $\sigma|h$ the continuation strategy profile induced by history h.

1.3 Equilibrium

To close the model, we must specify how the cut-off price y and the reservation price z are determined. In equilibrium, the strategy profile σ induces a distribution of prices that matches the understanding held by consumers in each t, F_t, and is consistent with the consumer reservation price, z_t, according to (3). Given the memory parameter m, (2) determines the cut-off price y_t. In Section 3 we initialise $y_0 = \infty$, while in Section 2 we abstract from initial conditions.

Given a sequence of consumer reservation prices $\{z_t\}_{t=0}^{\infty}$, payoffs for firm j are given by

$$V^j\left(\sigma, \{z_t\}_{t=0}^{\infty}\right) = \sum_{t=0}^{\infty} \delta^t \pi^j\left(a_t(\sigma), y_t(\sigma), z_t\right). \tag{8}$$

Equilibrium is defined formally below. We focus on market equilibria that are optimal for the cartel.

Definition 1. The collection $\{\sigma, \{y_t, z_t, F_t\}_{t=0}^{\infty}\}$ is a *market equilibrium* if:
(1) given any history $h \in \mathscr{H}$, $V^j(\sigma|h, \{z_t\}_{t=0}^{\infty}) \geq V^j((\sigma'^j;\sigma^{-j})|h, \{z_t\}_{t=0}^{\infty})$ for all j and σ'^j; and
(2) for all t, the cut-off price y_t is given by (2), the distribution F_t is induced by $a_t(\sigma)$, and the consumer reservation price z_t is determined by (3).

In Sections 2 and 3, we consider pure symmetric strategies. In Section 4.2, we discuss asymmetric and mixed strategies. As we shall see, Definition 1 gives rise to an analytic solution for optimal cartel pricing policies. It is worth noting that qualitative features of this solution survive adjustments to this setup. For example, in Section 2.2, we discuss the role of F_t and argue alternative mechanisms for consumer price understanding are possible; and in Section 4.1, we show that our results are robust to alternative forms of (2) and (1).

[20] Strictly, the firms play a dynamic game with state vector given by the most recent m market minimum prices (see (2)). Because the state is a subset of the current history, we economise on notation by omitting the dependence of strategies on the state.

2 Infinitely repeated cycles

To aid exposition, we temporarily consider two methodological shortcuts which we resolve in Section 3. First, we consider equilibria in a restricted class of strategies in which firms adopt infinitely repeated symmetric k-period sequences of prices with $k \le m$, defined below. The strategy $\sigma^{(k)}$ is a variation of the grim-trigger strategy allowing for a time-varying equilibrium path. On the equilibrium path, each firm repeats the finite sequence $\{p_1, p_2, \dots, p_k\}$.

Definition 2. In a *k-period cycle* $\sigma^{(k)}$, in period t firm j sets price

$$p_t^j = \begin{cases} p(t) & \text{if } t = 0 \text{ or } p_\tau^i = p(\tau)\ \forall\, \tau < t,\ \forall\, i; \\ 0 & \text{otherwise,} \end{cases} \tag{9}$$

where

$$p(t) = p_s \text{ if } t - s + 1 \text{ is divisible by } k,\ \ s = 1, \dots, k, \tag{10}$$

and $p_s \in [0,1]$, $s = 1, \dots, k$.

Second, we abstract from the initial k periods of play and focus on a cartel with a mature history. With cyclical strategies $\sigma^{(k)}$ and $k \le m$, this allows us to consider a constant cutoff, $y_t = \min_s p_s$ for all t. It then follows directly from (1) that punishment under (9) represents an optimal penal code with value 0.

A special case of $\sigma^{(k)}$ is a constant price path in which $p_s = p$, $s = 1, \dots, k$, for constant p. As a benchmark, the following lemma handles this case.

Lemma 2. *There exists a market equilibrium in which firms play* $\sigma^{(k)}$ *with* $p_s = p \in (0,1]$ *for* $s = 1, \dots, k$ *if and only if* $\delta \ge (n-1)/n$.

To consider strategies with time-varying prices, we introduce additional notation. Where the context is clear, subscripts refer to periods within a k-period cycle, rather than time. Define v_s, $s = 1, \dots, k$, to be the continuation value for each cartel member starting from period s of the strategy $\sigma^{(k)}$:

$$v_1 = \frac{\pi_1 + \delta\pi_2 + \cdots + \delta^{k-1}\pi_k}{1-\delta^k}, v_2 = \frac{\pi_2 + \delta\pi_3 + \cdots + \delta^{k-1}\pi_1}{1-\delta^k}, \dots, v_k = \frac{\pi_k + \delta\pi_1 + \cdots + \delta^{k-1}\pi_{k-1}}{1-\delta^k},$$

where $\pi_s \equiv \pi^j(a_s(\sigma^{(k)}), y_s(\sigma^{(k)}), z_s) = p_s/n$, and notice that

$$v_1 = \delta v_2 + p_1/n,\ \ v_2 = \delta v_3 + p_2/n,\ \ \dots,\ \ v_k = \delta v_1 + p_k/n. \tag{11}$$

Assign p_k to be the lowest point in the firm's price sequence:

$$p_k = \min_{s \in \{1,\dots,k\}} p_s,$$

and define v as follows:

$$v = \max\{v_1, \dots, v_k\}.$$

Notice that if v is the cartel's objective, assigning the minimum point of the cycle to position k in the price sequence is without loss of generality. That is, it is equivalent to considering v_1 as the cartel objective and allowing the position of the minimum price to be unrestricted.

We now set up the cartel's problem through Lemma 3.

Lemma 3. *Suppose $\delta < (n-1)/n$. Then, for $k \leq m$, the optimal k-period cycle $\sigma^{(k)}$ consistent with market equilibrium is given by the solution to the following program:*

$$\max_{p_1,\dots,p_k \in [0,1]} v \tag{12}$$

subject to

$$v_s \geq \alpha_1 p_s, \ s = 1, \dots, k-1, \ \textit{and} \ v_k \geq p_k, \tag{13}$$

$$v_s \geq p_k, \ s = 1, \dots, k-1, \tag{14}$$

$$v_s \geq \alpha_n \min\{1, z_s\}, \ s = 1, \dots, k-1. \tag{15}$$

With $k \leq m$, limited consumer memory plays no role and the cut-off price is time-invariant, $y_s = p_k$, $s = 1, \dots, k$. The cartel maximises the discounted value of the joint profit stream subject to three sets of incentive compatibility constraints that correspond to three kinds of deviation. First, suppose firm j considers marginally undercutting the current prescribed price. For any period $s < k$ of the cycle in which $p_s > p_k$, this deviation is above the lowest price in the cycle, $y_s = p_k$. Deviation therefore yields market share α_1 and profit $\alpha_1 p_s$. In period k, marginal undercutting is below $y_k = p_k$ and is observed by all consumers, leading to profits p_k. The constraints in (13) therefore deter any deviation of this type. Second, according to (5), deeper price cuts do not yield a larger market share unless firm j sets a price below p_k. The constraints in (14) prevent this form of deviation. Third, firm j also considers deviating by raising price. In period k, $p_k = y_k$, and an upward deviation is ineffective. In any other period, any price between the current cartel price and the consumer reservation price leads to market share α_n. Therefore, the most profitable price rise in any period $s \leq k-1$ involves $p = \min\{1, z_s\}$. The constraints in (15) deter this deviation. For future reference, number the constraints in (13) by $1, \dots, k$, where constraint s involves v_s.

With Lemma 3 in hand, we need only solve the program (12) - (15) to identify a market equilibrium that is jointly optimal for firms. We consider non-trivial solutions with a strictly positive cartel value, $v > 0$. The nature of the problem faced by the cartel depends on the nature of deviations that must be guarded against. In high salience environments (i.e. "high" β environments), deviations are more often observed and recalled by consumers, and the greatest danger to the cartel is the temptation to undercut. We deal with this case in Section 2.1 below. Alternatively, if salience is low and search costs are high, another threat to cartel discipline arises from the temptation to raise price above the prescribed level, secure in the understanding that consumers are unlikely to notice that a deviator's price is unusually high. We turn to this case in Section 2.2.

2.1 High salience environments

In this section we characterise the optimal cartel price path for parameter vector (β, δ, n, k, c) if the salience parameter β is sufficiently high, as we later define in Lemma 8. The workhorse of the paper is Lemma 4, in which we solve for the optimal k-period cycle that is impervious to undercutting. The remaining results provide additional detail for our solution. Lemma 5 demarcates ranges of values of the discount factor that give rise to equilibria with different properties. Lemma 6 then provides an explicit solution to the cartel's optimal price path.

We first define the critical discount factor

$$\delta_1(\beta, n, k) \equiv \left(\frac{\alpha_1 n - 1}{\alpha_1 n}\right)^{\frac{k-1}{k}} \left(\frac{n-1}{n}\right)^{\frac{1}{k}}. \tag{16}$$

As we show in Lemma 4, δ_1 is the lowest discount factor amenable to collusive equilibria in high salience environments.

Lemma 4. *For $k \le m$, there exists a k-period cycle $\sigma^{(k)}$ that solves the program (12) - (14) if and only if $\delta \ge \delta_1$. The cycle is unique. If $\delta \in \left[\delta_1, \frac{n-1}{n}\right)$, the cycle has the following properties:*
(i) prices decline monotonically over the cycle: $p_1 \ge \cdots \ge p_{k-1} > p_k$, with $p_1 = 1$;
(ii) $p_s = \min\left\{1, \frac{\delta \alpha_1 n}{\alpha_1 n - 1} p_{s+1}\right\}$, $s = 2, \ldots, k-2$, $p_{k-1} = \min\left\{1, \frac{\delta n}{\alpha_1 n - 1} p_k\right\}$, and $p_k = \frac{\delta n}{n-1} v_1$;
(iii) $v = v_1$ is an increasing function of δ;
(iv) $v_s = \min\{1/n + \delta v_{s+1}, \frac{\delta \alpha_1 n}{\alpha_1 n - 1} v_{s+1}\}$, $s = 1, \ldots, k-1$, and $v_k = \frac{\delta n}{n-1} v_1$.

An implication of Lemma 4 is that p_s is increasing in δ for $s = 1, \ldots, k$. This allows us to offer the following definitions, which are useful for our discussion of the Lemma.

Definition 3. The *knot discount factor* δ_i connects two regions: if $\delta < \delta_i$ the equilibrium path has $p_i < 1$; if $\delta \ge \delta_i$ the equilibrium path has $p_i = 1$, for $i = 2, \ldots, k-1$.

Definition 4. i) An equilibrium price path is a *pure sales* path if $p_s = 1 > p_k$ for $s = 1, \dots, k-1$. ii) An equilibrium price path is a *distinct cycle* path if $p_s < 1$ for $s = 2, \dots, k-1$.

Lemma 4 implies that there is a range of discount factors, $[\delta_1, (n-1)/n)$, for which collusion is sustainable with a dispersed price path, but not with a fixed price. In this range, firms optimally coordinate on dispersed prices to reduce the visibility of any potential deviation, thereby relaxing the incentive constraints of the cartel. If β (and hence α_1) is smaller, consumers have greater difficulty penetrating the haze of price dispersion and the cartel is sustainable for a greater range of discount factors. Intuitively, if consumers have greater difficulty recalling specific prices that are within their realm of experience, then intertemporally dispersed prices provide a greater shield against deviation by undercutting. Notice also that in the limit, as consumer attentiveness approaches the ideal, the sustainability of collusive price paths converges towards the perfect information environment. That is, δ_1 converges towards $(n-1)/n$ as α_1 approaches 1.

The optimal price path is described by a set of complementary slackness conditions. If constraint s is not binding, firms set the choke price; if it is binding, prices must decline to uphold the incentive constraints in (13). Because p_k is the lowest price, a firm undercutting in period k receives a market share of 1. Therefore constraint k is the most difficult to satisfy, and it is always binding for $\delta < (n-1)/n$. For $\delta \in [\delta_{k-1}, \delta_k)$, constraint k is the only binding constraint, and a pure sales path is observed. Sales obfuscate the price process in the eyes of consumers by generating a range of observed prices, and achieve this at minimum cost. That is, firms can charge the choke price in every period of the cycle except the last.

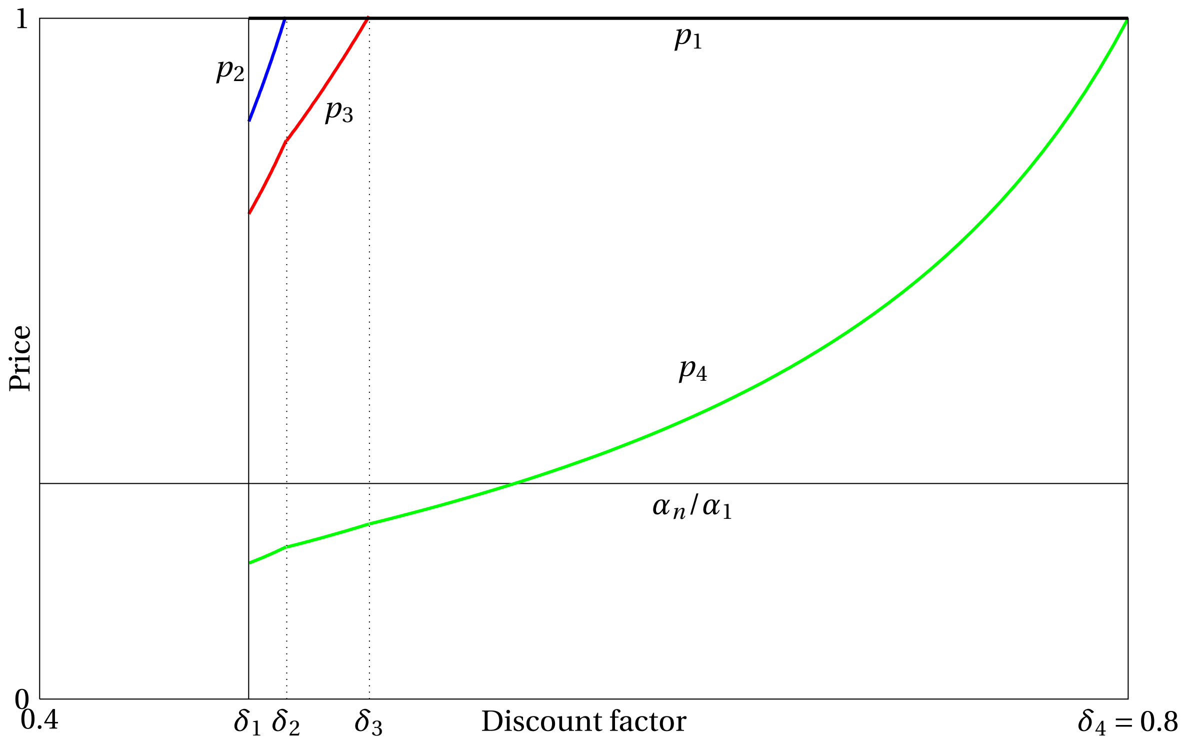
For lower values of δ, p_k must be reduced to satisfy the binding constraint k. Within the region $[\delta_{k-1}, \delta_k)$, the other constraints in (13) remain slack. At $\delta = \delta_{k-1}$, constraint $k-1$ becomes binding, and this constraint can only be satisfied in the face of a lower discount factor by lowering p_{k-1}. As the discount factor is reduced below successive knot discount factors, successive prices must be lowered to satisfy the incentive constraints. In the limit of this process, all prices except the first in the sequence must be lowered. A distinct cycle path occurs for $\delta \in [\delta_1, \delta_2)$.

The following two Lemmas complete our characterisation of equilibrium in high salience environments. Lemma 5 describes the knot discount factors associated with the complementary slackness conditions in part (ii) of Lemma 4.

Lemma 5. *The knot discount factors $\delta_s, s = 2, \dots, k-1$, are determined by the following equations:*

$$\delta_s^k + \frac{1}{\alpha_1 n} \sum_{i=1}^{s-1} \delta_s^{k-i} = \left(\frac{\alpha_1 n - 1}{\alpha_1 n} \right)^{k-s} \left(\frac{n-1}{n} \right). \tag{17}$$

Figure 3: Equilibrium price paths



Part (ii) of Lemma 4 implies that the optimal price path is completely determined by the price in the last period of the cycle, p_k. We solve for this in Lemma 6. This permits an explicit solution for the optimal price path and accompanying value of the objective function.

Lemma 6. *For $s = 1, \ldots, k-1$, if $\delta \in [\delta_s, \delta_{s+1})$, then a k-period cycle solving the program (12) - (14) yields*

$$v_1 = \frac{(n-1)(1-\delta^s)}{n(1-\delta)\left(n\left(1-\left(\frac{\alpha_1 n}{\alpha_1 n-1}\right)^{k-s-1}\delta^k\right)-1\right)}, \quad p_k = \frac{\delta(1-\delta^s)}{\left(n\left(1-\left(\frac{\alpha_1 n}{\alpha_1 n-1}\right)^{k-s-1}\delta^k\right)-1\right)(1-\delta)}, \tag{18}$$

where v_1 and p_k are continuous in δ for $\delta \in [\delta_s, \delta_{s+1})$.

Together, Lemmas 4 - 6 constitute a complete description of equilibrium in high salience environments. Example 1 illustrates the equilibrium.

Example 1. There are $n = 5$ firms who pursue strategies with cycle length $k = 4$. With $\beta = 0.3$, (5) and (4) lead to market shares $\alpha_1 = 0.334$ and $\alpha_n = 0.106$. Figure 3 depicts the resulting equilibrium price path for a range of possible discount factors. Price is on the vertical axis and the horizontal axis indexes discount factors. For a given discount factor, we can read the picture vertically to reveal prices at each point of the cycle. Recall that $p_1 = 1$ and that each knot discount factor δ_s delineates regions where $p_s = 1$ and $p_s < 1$. The knot discount factors are indicated with $\delta_1 \approx 0.477$, $\delta_2 \approx 0.491$, $\delta_3 \approx 0.521$, and $\delta_4 = 0.8$. For $\delta > \delta_4 = 0.8$, a constant price path is sustainable. A price cycle equilibrium exists for $\delta \in [\delta_1, \delta_4)$. For $\delta \in [\delta_3, \delta_4)$ we observe a pure sales path. As δ falls, p_4 falls, until we hit the knot discount factor δ_3, at which point, p_3 begins falling as well. For $\delta \in [\delta_2, \delta_3)$ we then observe more extensive sales with two periods of monopoly pricing in each cycle and two periods of discounted prices in between. Finally, in the region $\delta \in [\delta_1, \delta_2)$, we observe distinct cycle paths. For our chosen parameters, the constraints in (15) play no role.

Figure 4 illustrates the path of prices for different discount factors. In each panel, prices are shown on the vertical axis and the horizontal axis indexes time. $\delta = \delta_1$ is the lowest discount factor conducive to a collusive equilibrium, and this case is shown in the top left panel. We observe a distinct cycle path, with prices varying between 1 and approximately 0.2 in an asymmetric cycle. With the higher discount factor of $\delta = \delta_2$ shown in the top right panel, a more profitable path is sustainable. We can see that constraint 2 is now slack, permitting a 2-period spell of monopoly pricing, and a slightly higher cycle minimum of $p_4 \approx 0.22$. The bottom left panel shows an example of a sales path with $\delta = \delta_3$. A regular sale is observed every 4th period. As the discount factor is raised further, the depth of the sale is reduced. In the limit (that is for $\delta \geq \delta_4$), no sale is required to maintain the incentives for cooperation, and a constant price path is observed. This case is shown in the bottom right panel. □

2.2 General salience environments

In Section 2.1, we solved the cartel's problem under the presumption that the only threat to cartel discipline came from the temptation to undercut. When salience is low and search costs are high, the temptation to deviate by raising price may also be a threat. A firm setting a higher price than her rivals loses fewer customers if consumers are less attentive. This increases the incentive to raise price, particularly when cartel policies call for a low price.

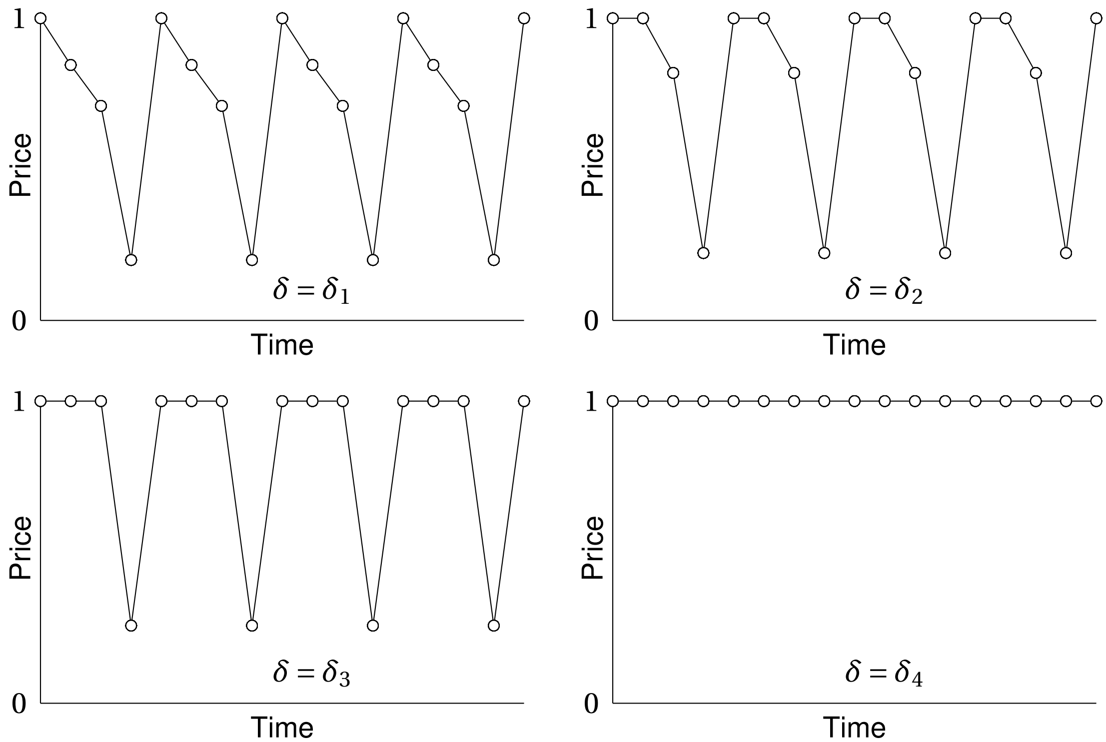
We first use Lemma 7 to clarify the role of search costs and consumer beliefs about the current price distribution, F_s. Let

$$\gamma(n,\beta,c) = \begin{cases} \alpha_n/\alpha_1 & \text{if } c \geq 1-\alpha_n; \\ \frac{\alpha_n c}{(1-\alpha_n)\alpha_1} & \text{if } c < 1-\alpha_n. \end{cases} \tag{19}$$

Lemma 7. *Suppose $\{p_s\}_{s=1}^k$ solve the program (12) -(14). Then the constraints in (15) are satisfied iff $p_{k-1} \geq \gamma$, where γ is defined in (19).*

Recall that in period k, $p_k = y_k$ and upward price deviations are ineffective. The most challenging relenting constraint for the cartel is in period $k-1$. Search costs influence the

Figure 4: Price cycles



relenting constraint by determining the optimal upward deviation. If c is sufficiently high, then for any price below the choke price, consumers will not search. The optimal upward deviation is therefore to the choke price and deviation profits are given by α_n. For lower values of c, the optimal upward deviation is to the reservation price, $z_s(c)$. Because the reservation price is increasing in c, lower search costs actually aid collusion in low-salience settings by reducing the profits of upward price deviations.

If F_s reflects the current price distribution induced by $\sigma^{(k)}$ in period s, then $z_s(c) = p_s + c$, giving rise to the specific form of (19). Alternative specifications of F_s are possible. For example, F_s may depend on a consumer's price observations over her finite memory. We can trace through the implications of alternative definitions by adjusting $z_s(c)$.

Lemma 8 describes the optimal price path for general salience environments.

Lemma 8. *Suppose* $\{p_s\}_{s=1}^k$ *solve the program (12) -(14).*
i) There exists $\beta_1(n,k,c)$ *such that for any* $\beta \geq \beta_1$, $\{p_s\}_{s=1}^k$ *also satisfy the constraints in (15), where* β_1 *solves*

$$\gamma(n,\beta_1,c) = p_{k-1}(\beta_1,\delta_1(\beta_1)). \tag{20}$$

ii) If $\beta < \beta_1$*, then there exists* $\hat{\delta}(\beta,n,k,c)$ *such that for* $\delta < \hat{\delta}$*, there is no solution to the program*

(12) - (15) with positive prices, while for $\delta \geq \hat{\delta}$, $\{p_s\}_{s=1}^{k}$ also solve the program (12) - (15). In particular, $\hat{\delta}$ solves

$$\gamma(n,\beta,c) = p_{k-1}(\beta,\hat{\delta}). \tag{21}$$

iii) For any $\beta > 0$, there exists $\overline{c} > 0$ such that $\forall\ c < \overline{c}$, $\{p_s\}_{s=1}^{k}$ also solve the program (12) - (15).

In the presence of price dispersion, the attentiveness of consumers impacts the profitability of deviation in two ways. For low values of β, consumers may not notice the attractiveness of a low-priced firm, reducing the payoffs to undercutting the cartel price. In addition, a high-priced firm may still attract some custom, increasing the profitability of deviations that involve price rises. As the value of β rises, both factors increase the attractiveness of undercutting deviations relative to relenting deviations. This feature allows us to divide the parameter space into a high-attentiveness region in which the cartel is primarily concerned with undercutting deviations, and a low-attentiveness region in which relenting is also a concern.

Part i) of Lemma 8 suggests that, for given values of (n,k,c), if β is above a critical value, raising price is never a profitable deviation. In Part ii) we argue that if consumer attentiveness does not meet this threshold, collusive profits can still be salvaged if firms are sufficiently patient. Further, if the relenting constraints (15) are met, the optimal price path is entirely unaffected by them. In Part iii), we highlight the role of search costs. Because search costs do not enter the undercutting constraints, they have no impact on the shape of the optimal path. Instead, they influence the sustainability of the optimal path through the relenting constraints. For low c, the reservation price is close to the prescribed cartel price and the value of a relenting deviation is minimal. The relenting constraints can then be ignored if c is sufficiently low.

Lemma 8 brings into focus the contrasting roles of the three sets of incentive compatibility constraints. The constraints in (13) prevent marginal undercutting. These constraints determine the shape of the price path and, in high salience settings, also determine the sustainability of the cartel. The constraints in (14) ensure firms have no incentive to undercut below the cut-off price. These constraints are always satisfied when (13) holds. We can see this by comparing the two sets of constraints. An optimal price path that satisfies (13) yields a cartel continuation value that decreases over time within each price sequence. Therefore, if constraint k in (13) is satisfied, then all constraints in (14) will also hold.

Finally, the relenting constraints in (15) play a binary role. In markets with attentive consumers, these constraints can be ignored. With inattentive consumers, these constraints determine the sustainability of collusion, but have no effect on the shape of the optimal price path if collusion is sustainable. This last result arises from a direct conflict between the two sets of constraints (13) and (15). To ease concerns about undercutting, the cartel must lower prices to reduce the current payoffs to deviation. By contrast, to mitigate the incentive to relent, the cartel must raise prices in order to raise the continuation value of the cartel relative to the monopoly profit that a deviator would receive. An optimal cartel seeks to maximise cartel value. Therefore, prices will be maximised subject to the marginal undercutting constraints in (13). If the constraints in (15) are also satisfied, then collusion is feasible. If they are not, then there is no further scope to raise prices.

The following example illustrates the determination of the optimal price path in low-salience environments.

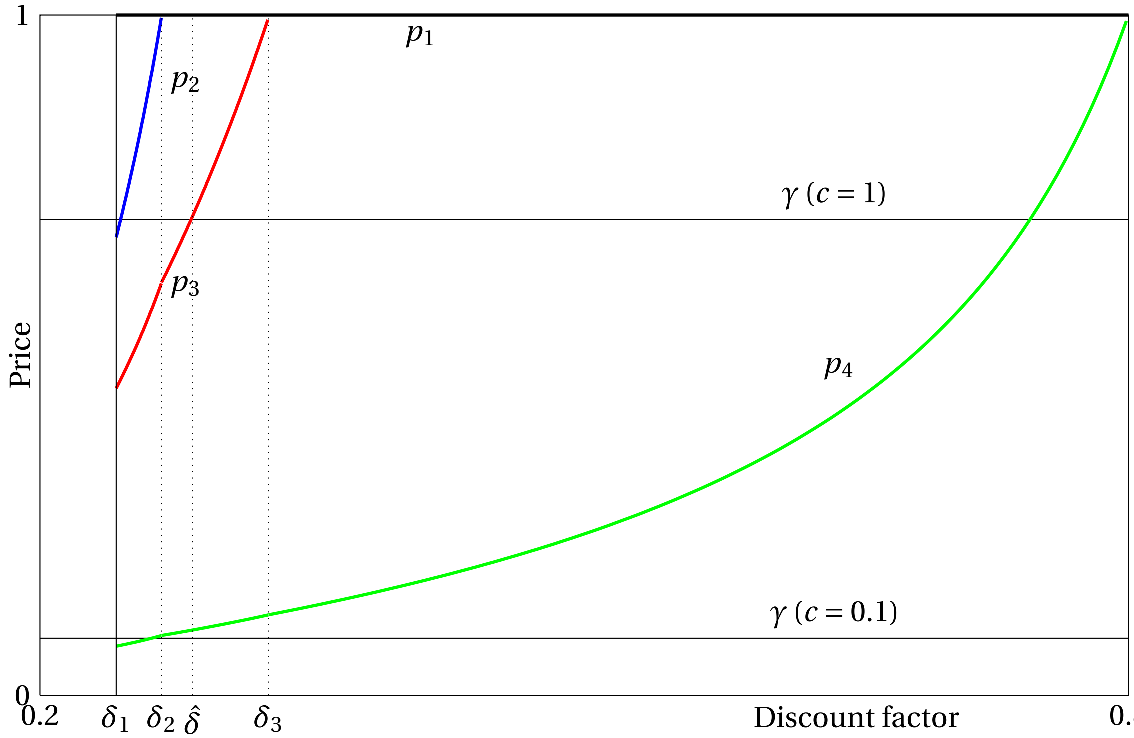
Example 2. As in Example 1, there are $n = 5$ firms who choose an optimal cycle of length $k = 4$. We set $\beta = 0.1$, leading to market shares $\alpha_1 = 0.218$ and $\alpha_n = 0.184$. Figure 5 depicts the resulting equilibrium price path for a range of possible discount factors. The vertical axis describes price and the horizontal axis indexes discount factors. For a given discount factor, we read the picture vertically to show prices at each point of the cycle. For high search costs ($c \geq 1 - \alpha_n$), $\gamma = \alpha_n$. This is illustrated by the upper horizontal line. The intersection of γ and p_3 determines the sustainability of collusion; equilibrium exists only to the right of this intersection. Lower search costs lead to a reduced payoff to upward price deviations as suggested by the lower horizontal γ line. Thus, collusion is sustainable for a greater range of discount factors when search costs are lower. □

Lemma 8 implies that collusion requires cartel vigilance towards both undercutting and relenting. A sufficiently patient cartel can overcome both obstacles. For $\beta \geq \beta_1$, Lemmas 4 and 8 demonstrate that collusion is sustainable if and only if $\delta \geq \delta_1$. For $\beta < \beta_1$, collusion is sustainable for $\delta \geq \hat{\delta}$ where $\hat{\delta}$ is determined by (21). Combining these results, collusion is sustainable if and only if $\delta \geq \delta^*$ where the critical discount factor δ^* is defined as

$$\delta^*(\beta, n, k, c) = \begin{cases} \delta_1(\beta, n, k) & \text{if } \beta \geq \beta_1; \\ \hat{\delta}(\beta, n, k, c) & \text{if } \beta < \beta_1. \end{cases} \tag{22}$$

Following on from our discussion of Lemma 4, let us reconsider comparative statics with respect to β. By Lemma 1, as β converges to 0, α_1 approaches $1/n$; with extreme consumer inattentiveness, the market share benefits of undercutting are minimal. This is reflected in δ_1, which converges to 0 as α_1 converges to $1/n$. For low values of β and high search costs, the cartel may also be vulnerable to deviations involving price rises. The constraint $\delta \geq \hat{\delta}$ applies in these cases. However, if both c and β are arbitrarily small, then the critical discount factor δ^* could also be arbitrarily small.

Figure 5: Price paths with low attentiveness



When $\beta < \beta_1$, the salience parameter β has qualitative implications for the shape of the price path. The following result demarcates values of β for which alternative shapes are optimal.

Lemma 9. *Suppose the prices* $\{p_i\}_{i=1}^{k}$ *solve the program (12) - (14). Then, for* $s = 1, \ldots, k-2$, *there exists* $\beta_s(n,k,c)$ *such that if* $\beta > \beta_s$, *the constraints in (15) are satisfied if* $\delta \geq \delta_s$.

Recall that the knot discount factors δ_s determine the shape of the cycle in terms of the number of distinct prices observed on the equilibrium path. Lemma 9 places restrictions on which of these cycle shapes could be optimal depending on β. Taking Lemmas 8 and 9 together, we can summarise the effect of the constraints (15) on the cartel's program for different salience parameters. If salience is sufficiently high ($\beta \geq \beta_1$), then these constraints are always satisfied, and we can rely on the solution we discussed in Section 2.1. For $\beta \in (\beta_2, \beta_1)$, there is a range of discount factors for which the distinct cycle equilibria are impacted by (15). At $\beta \leq \beta_2$, no distinct cycle paths satisfy these constraints optimally. For $\beta \leq \beta_s$, all optimal paths involve at most $k-s+1$ distinct prices. For $\beta \leq \beta_{k-1}$, the only optimal equilibria that survive these constraints are sales paths.

To illustrate how this process works, re-examine Example 2 as depicted in Figure 5. β_1 is determined by the intersection of γ and p_3. If $c = 0.1$, γ intersects p_3 for $\delta < \delta_1$. For all δ, equilibria are unaffected by the constraints (15). When $c = 1$, γ intersects p_3 for $\delta > \delta_3$. No equilibria to the program (12) - (15) exist for $\delta < \hat{\delta}$. The only optimal equilibria that remain (for $\delta \geq \hat{\delta}$) involve pure sales paths.

We illustrate the manner in which β influences cartel sustainability and the shape of the price path with the following example.

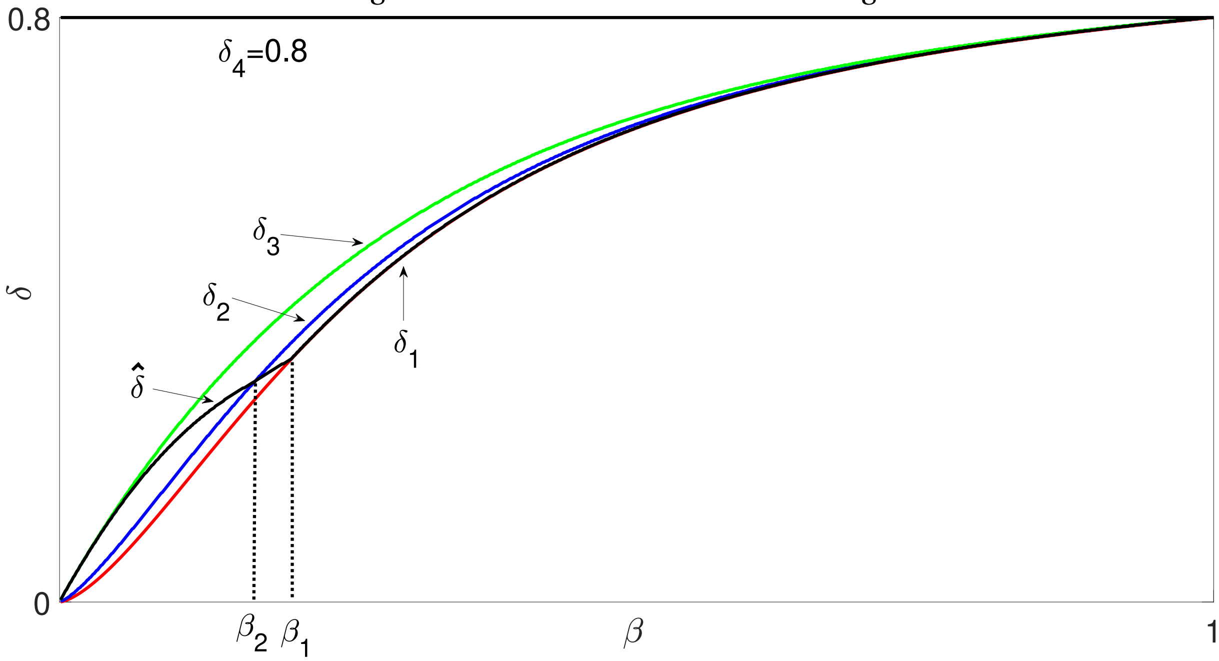
Example 3. Reconsider the setting of Examples 1 and 2 with $n = 5$, $k = 4$ and $c = 1$, and consider all $\beta \in [0,1]$. Figure 6 illustrates the determination of critical discount factors. The vertical axis depicts the discount factor and the horizontal axis indexes the salience parameter β. We can read this figure vertically. Fixing a particular value of β determines the knot discount factors δ_s and the critical value of the discount factor. Equilibrium exists if and only if $\delta \geq \delta^*$ as defined in (22). The line $\hat{\delta}$ delineates discount factors consistent with the relenting constraints (15).

We can use the example to illustrate comparative statics with respect to attentiveness, β. If $\beta \geq \beta_1$, then $\delta \geq \delta_1$ is required and cartel sustainability becomes harder for higher values of β. That is, δ_1 is increasing in β and δ_1 converges to $(n-1)/n$ as β approaches 1. For $\beta < \beta_1$, $\hat{\delta} > \delta_1$ and the relenting constraints in (15) determine the critical discount factor $\hat{\delta}$. Observe also that we must have $\hat{\delta} \leq \delta_3$. This can be seen from Figure 5: $\hat{\delta}$ is determined by the condition $p_3 = \alpha_n/\alpha_1$ (as per Lemma 8), and δ_3 is determined by the condition $p_3 = 1$ (as per Lemma 4). As β approaches zero, the ratio α_n/α_1 approaches 1, and $\hat{\delta}$ approaches δ_3 from below. Note that both δ_1 and δ_3 approach zero with β. Because $\hat{\delta}$ is sandwiched between these knot discount factors, the critical discount factor must also approach zero. Thus, the cartel becomes arbitrarily easy to sustain as attentiveness dissipates. □

We close this section by collecting our main results into the following proposition. Part (a) describes the requirements for existence of a non-trivial collusive equilibrium; Part (b) discusses the role of search costs; and Part (c) characterises the resultant equilibria.

Proposition 1. *(a) For $k \leq m$, there exists a market equilibrium with positive payoffs using strategies $\sigma^{(k)}$ if and only if $\delta \geq \delta^*(\beta, n, k, c)$ as defined in (16), (19), (21), and (22).*
(b) For any $\beta > 0$, there exists $\overline{c} > 0$ such that $\forall c < \overline{c}$, $\delta^*(\beta, n, k, c) = \delta_1(\beta, n, k)$.
(c) An optimal market equilibrium with the strategies $\sigma^{(k)}$ has the following price path:
(i) if $\delta \geq (n-1)/n$, then $p_s = 1$ for $s = 1, \ldots, k$;
(ii) if $\delta^* \leq \delta < (n-1)/n$, then p_k is given by (18), $p_{k-1} = \min\left\{1, \frac{\delta n}{\alpha n - 1} p_k\right\}$, $p_s = \min\left\{1, \frac{\delta \alpha n}{\alpha n - 1} p_{s+1}\right\}$ for $s = 2, \ldots, k-2$, and $p_1 = 1$.

Figure 6: Attentiveness and discounting



3 Finite memory

In this section, we relax the restriction to cyclical strategies and consider the initial conditions of the market. In Proposition 2, we show that cyclical strategies are optimal when consumer memory is finite. Let the optimal k period cycle $\sigma^{(k)}$ be defined by the equilibrium path price sequence $\{p_s\}_{s=1}^{k}$, as specified in Proposition 1, and define $\tilde{\sigma}^{(k)}$ analogously by the reconfigured sequence $\{p_k, p_1, p_2, \ldots, p_{k-1}\}$.

Proposition 2. *Suppose consumers have m-period memory and $y_0 = \infty$. Then*
(a) there exists a market equilibrium with positive payoffs if and only if $\delta \geq \delta^*(\beta, n, m+1, c)$;
(b) the optimal market equilibrium has the following properties:
(i) if $\delta \geq (n-1)/n$, then $p_s = 1$ for all s;
(ii) if $\delta^*(\beta, n, m+1, c) \leq \delta < (n-1)/n$, then $\tilde{\sigma}^{(k)}$ are optimal market equilibrium strategies, where $k = m+1$.

We use the following definition in the proof of Proposition 2. Condition 1) ensures that the price p_q is the lowest in memory in period s, and condition 2) implies that without obfuscation there would be an incentive to deviate in period s.

Definition 5. Given prices p_s and p_q, we say that
1) p_q *supports* p_s if $y_s = p_q$; and

2) p_s *relies on* p_q if $y_s = p_q$ and $v_s < p_s$.
A price is said to be *supporting* if it supports at least one other price.

According to Proposition 2, the optimal path is an infinitely repeated $m+1$ period cycle. Let us explain the intuition behind this result. Obfuscation allows the cartel to set higher prices by relaxing the constraints arising from the incentive to marginally undercut the cartel price. With finite consumer memory, periodic low supporting prices are required to provide obfuscation. For an optimal path, any price that does not play a supporting role must be maximised subject to the undercutting constraint. This ensures complementary slackness conditions mirroring those of Lemma 4, leading to a decreasing trajectory of prices between supporting prices.

Partition the optimal path into a sequence of subpaths defined as follows. For any two consecutive supporting prices p_r and p_s, define the subpath $\{p_r, \ldots, p_{s-1}\}$. We show that supporting prices are not reliant, conferring a degree of autonomy to each subpath. It follows that the value of a path can be improved while maintaining all incentive constraints by replacing less profitable subpaths with more profitable ones. Optimality then dictates that each subpath must be identical. This implies that the optimal path is an infinitely repeated cycle.

Longer cycles are preferable because they afford obfuscation protection and therefore permit higher prices for a greater fraction of the path. With the exception of periods containing a supporting price, every price in a cycle requires the supporting price to be in memory. The optimal cycle is therefore of length $m+1$ periods.

Finally, to deal with the initial condition $y_0 = \infty$, the initial price is supporting. The cycle therefore kicks off with the lowest price, establishing obfuscation protection for the remaining prices.

4 Discussion and extensions

In this section we discuss generalisations of our model in several dimensions. We begin by arguing, in Section 4.1, that our qualitative conclusions survive adjustments in the salience relation (1). In Section 4.2, we ask whether strategies involving contemporaneous price dispersion could improve the value of the cartel. Section 4.3 examines the implications of forward-looking consumers. Finally, in Section 4.4, we sketch potential explanations for observed price dynamics in two prominent markets.

4.1 Consumer attentiveness

In Section 2, we learned that the determinants of the optimal price path are discounting and the profits from undercutting the current cartel price. As long as deviation payoffs increase with the cartel price, the essential features of the path will be retained. We illustrate with three generalisations of the salience relation (1).

First, note that discontinuity in (1) is not required. For example, suppose that salience transitions linearly between 1 and β in the neighbourhood $[y, y+\epsilon]$. Then we can show that market equilibrium is unaffected as long as the transition is sufficiently fast (i.e. ϵ is sufficiently small).

Second, consider an attentiveness process in which consumers are more likely to recall lower prices in a more graduated manner. Given a decreasing price path $\{p_s\}_{s=1}^k$ with $k \leq m$,[21] define the cut-off price levels $\{y_s\}_{s=1}^k$ by $y_s = p_s$, $s = 1, \ldots, k$, and redefine

$$\phi\left(p, \{y_s\}_{s=1}^k\right) = \begin{cases} \beta^{(s)} & \text{if } p \in (y_{s+1}, y_s],\ s = 1, \ldots, k-1; \\ 1 & \text{if } p \leq y_k, \end{cases} \tag{23}$$

where $\beta^{(s)} \in (0,1)$ increases in s for $s = 1, \ldots, k-1$. Thus, consumers are more likely to recall a specific price p the less often prices of that level or lower are observed.

We examine optimal symmetric equilibria. Given (23) and a strictly decreasing price path $\{p_s\}_{s=1}^k$ with $p_s \leq z_s$ for $s = 1, \ldots, k$, if firm j deviates from the price path in period s by setting $p_r \leq z_s$ instead of p_s, she obtains the market share

$$\alpha_i^{(r,s)} = \beta^{(r)}(1-\beta^{(s)})^{i-1} + (1-\beta^{(r)})(1-\beta^{(s)})^{n-1}/n, \ \ r, s = 1, \ldots, k, \tag{24}$$

where $i = 1$ if $p_r < p_s$ and $i = n$ if $p_r > p_s$. Ties are resolved uniformly as in (5). We will write $\alpha_i^{(s)} = \alpha_i^{(s,s)}$.

We set up Proposition 3 with the following definitions. Define the price and value paths $\{p_s\}_{s=1}^k$ and $\{v_s\}_{s=1}^k$ by the following relationships:

$$p_1 = 1, \ \ p_s = \min\left\{1, \frac{\delta \alpha_1^{(s+1)} n}{\alpha_1^{(s)} n - 1} p_{s+1}\right\}, s = 2, \ldots, k-1, \ \ p_k = \frac{\delta n v_1}{n-1}; \tag{25}$$

$$v_s = \min\left\{1/n + \delta v_{s+1}, \frac{\delta \alpha_1^{(s)} n}{\alpha_1^{(s)} n - 1} v_{s+1}\right\}, s = 1, \ldots, k-1, \ \ v_k = \frac{\delta n v_1}{n-1}, \tag{26}$$

implicitly define the relenting constraints,

$$v_s \geq \alpha_n^{(r,s)} \min\{p_r, z_s\}, \ \ r < s, s = 1, \ldots, k, \tag{27}$$

[21] If $k > m$, the cut-off price levels become time-varying. This requires additional notation without providing additional insight.

and define the discount factors $\delta_1(\boldsymbol{\beta}, n, k)$, $\delta_a(\boldsymbol{\beta}, n, k)$, and $\delta_b(\boldsymbol{\beta}, n, k)$ by

$$\delta_1 = \prod_{s=1}^{k} \left(\frac{\alpha_1^{(s)} n - 1}{\alpha_1^{(s)} n} \right)^{\frac{1}{k}}, \qquad \delta_a = \frac{\alpha_1^{(k-1)} n - 1}{n}, \qquad \delta_b = \max_{s=1,\dots,k-2} \frac{\alpha_1^{(s)} n - 1}{\alpha_1^{(s)} n} \frac{\alpha_1^{(s+1,s)}}{\alpha_1^{(s+1)}}, \tag{28}$$

where $\boldsymbol{\beta} = (\beta_1, \dots, \beta_{k-1})$.

Proposition 3. *If $\delta \in [\max\{\delta_1, \delta_a, \delta_b\}, (n-1)/n)$ and $\{p_s\}_{s=1}^k$ and $\{v_s\}_{s=1}^k$ satisfy (27), then in the optimal market equilibrium, the price and value paths are determined by (25) and (26).*

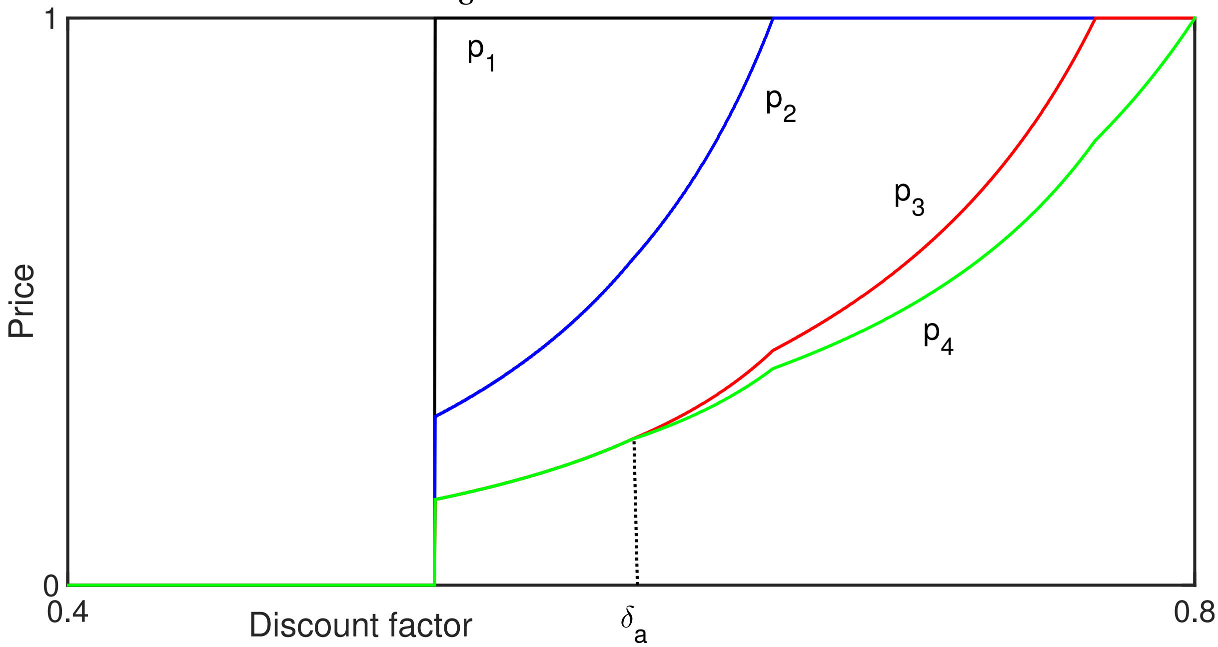
The condition $\delta \geq \max\{\delta_a, \delta_b\}$ ensures the price path (25) is decreasing. If the path is strictly decreasing, then deviating in period s to the price p_r with $r < s$ gives the market share $\alpha_n^{(r,s)}$. In the event that there exists $q > r$ with $p_r = p_q > p_s$, the constraint $v_s \geq \alpha_n^{(q,s)} \min\{p_q, z_s\}$ implies the constraint $v_s \geq \alpha_n^{(r,s)} \min\{p_r, z_s\}$. The constraints in (27) therefore deter all upward price deviations for all decreasing paths. These constraints can be satisfied, for example, by setting c sufficiently low.

By imposing (27), we focus attention on the incentive to undercut. With a decreasing path, for any $\delta \geq \delta_1$, there is no incentive to marginally undercut the cartel price. The expression for δ_1 in (28) is a natural generalisation of (16). With the salience relation (23), cartel members may also be tempted to undercut more aggressively to attract a higher market share. The condition $\delta \geq \delta_b$ is sufficient to prevent undercuts of this nature. This condition arises from comparing the incentive to marginally undercut the price p_s with the incentive to undercut by a full cut-off level to the price p_{s+1}. If $\delta \geq \delta_b$, then the period $s+1$ marginal undercutting constraint implies the "one-step" undercutting constraint in period s, for any s. With the market shares in (24), if a one-step undercut is deterred, then all q-step undercuts are also deterred, for $q > 1$.

Thus, for $\delta \geq \delta_b$, marginal undercutting constraints determine the shape of the optimal price path, and (25) and (26) suggest an optimal path that is closely related to that of the main model. For $\delta \in [\delta_1, \max\{\delta_a, \delta_b\})$, collusion may still be viable with a dispersed price path. However, the optimal path is impacted if either i) $\delta < \delta_a$ and monotonicity is violated by (25); or ii) $\delta < \delta_b$ and there is an incentive to undercut by at least one cut-off level with the path (25). Example 4 below provides an illustration of i). Example 8 in Section 4.4 illustrates ii).

Example 4 (Extended sale)**.** Let $n = 5$, $k = 4$, and $(\beta^{(1)}, \beta^{(2)}, \beta^{(3)}) = (0.2, 0.4, 0.8)$. Figure 7 illustrates the equilibrium price path for a range of discount factors. With these parameters, $\delta_a > \delta_1 > \delta_b$. For $\delta \geq \delta_a$, the optimal price path described by Proposition 3 is qualitatively similar to that of the main model (see Figure 3). The main difference is the proximity of the

Figure 7: An extended sale



p_3 and p_4 lines. With $\beta^{(3)}$ close to 1, obfuscation does not provide much protection in the range $(p_4, p_3]$. Examining (25), we see that the gap between p_3 and p_4 narrows for lower δ. For $\delta < \delta_a$, (25) violates the monotonicity of prices. The optimal path departs from (25) and dictates instead that $p_3 = p_4$, where the prices p_3 and p_4 are determined by the binding constraint $v_3 \geq p_{(1)}$. The equilibrium path then involves an extended sale in which the lowest price is offered for two consecutive periods. □

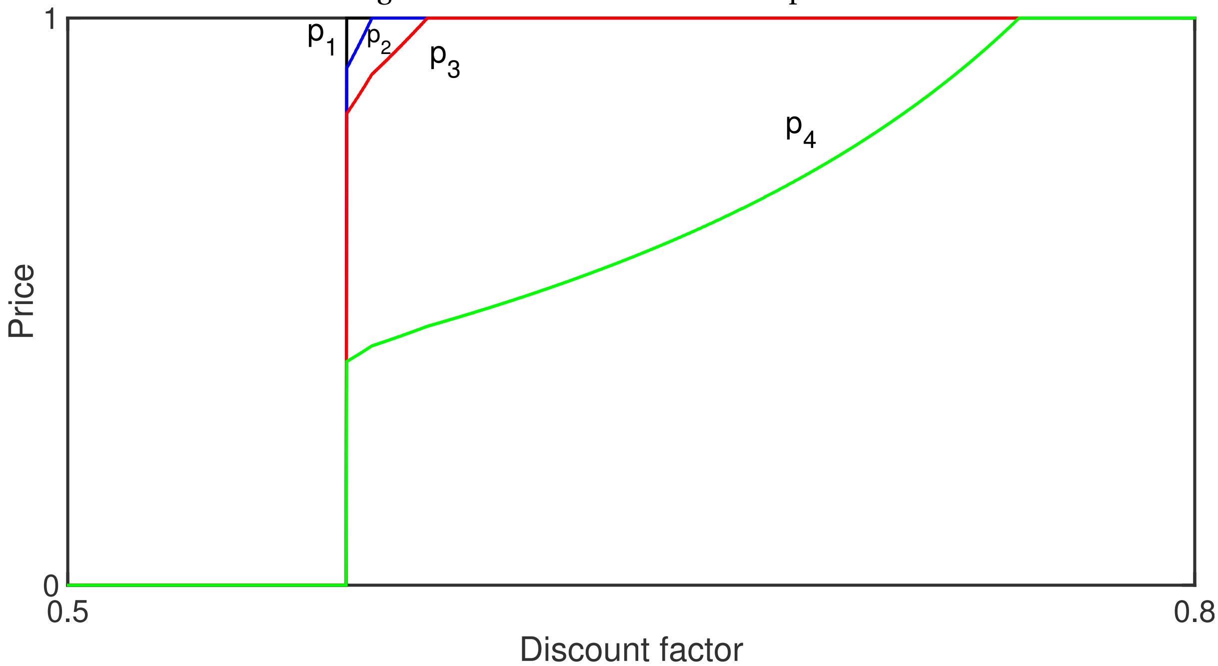
Third, we consider an alternative adjustment to the salience relation (1):

$$\phi(p, y) = \begin{cases} \beta^{(1)} & \text{if } p > y, \\ \beta^{(2)} & \text{if } p \leq y, \end{cases} \tag{29}$$

where $0 \leq \beta^{(1)} < \beta^{(2)} < 1$. Under (29), consumers do not remember unusually low prices for sure, but merely with a higher probability. We discuss the implications of this specification using the following example.

Example 5 (Inattention towards all prices)**.** Let $n = 5$, $k = 4$, and $(\beta^{(1)}, \beta^{(2)}) = (0.4, 0.8)$. Figure 8 illustrates the resulting price path. For $\beta^{(2)} < 1$, the profitability of deviation and punishment are both affected by limited attentiveness. Undercutting below the lowest price no longer delivers the entire market to the deviating firm. This reduces the profitability of undercutting at the bottom of the cycle. At the same time, a deviating firm can no longer be held

Figure 8: Inattention towards all prices



to a zero-value punishment regime.[22] Even if all rivals set a price of zero, positive profits are possible for a firm with a positive price. The sustainability of collusion with a fixed price is also affected. Collusion with a constant price of 1 is viable for $\delta \geq \frac{\alpha_1^{(2,1)} n - 1}{\alpha_1^{(2,1)} n - \alpha_n^{(1,2)} n} \approx 0.752 < \frac{n-1}{n}$.
□

4.2 Asymmetric and mixed strategies

We have restricted attention to strategies that are symmetric, coordinated, and pure. Two factors might limit the applicability of more general strategies. First, they complicate the cartel's problem by requiring coordination on the role of each member in addition to the price path itself. In the case of mixed strategies, it is also difficult to verify adherence to cartel strategies. Second, such paths introduce contemporaneous price dispersion, a feature absent from the main model. This presents an incentive for consumers to pay greater attention to prices by providing contemporaneous search benefits.

In this section, we discuss asymmetric strategies and then mixed strategies. For simplicity, we focus primarily on two-player games, and we maintain $c = 1$, effectively ruling out active search. Throughout, we define $\alpha = \alpha_1$ and $\gamma = \alpha_2$.

[22] In Appendix B, we present a detailed construction of the optimal penal code that covers this case.

4.2.1 Asymmetric strategies

Proposition 4 compares the use of symmetric and asymmetric pure strategies.

Proposition 4. *If $n = 2$ and $c = 1$, there is no contemporaneous price dispersion in the market equilibrium that maximises the average value of cartel members.*

The proof relies on the following argument. In periods that do not contain supporting prices, a price path involving contemporaneous price dispersion endows a higher market share to the lower priced firm and a positive share to the higher priced rival. Therefore, we can construct a symmetric price path of greater aggregate value by adopting the average price in these periods. In periods with supporting prices, we assign the minimum price in the current period to the symmetric price path. Because a high-priced firm receives no market share in these periods, this leads to no loss in aggregate value. We show that all the required incentive constraints are satisfied with this symmetric price path, leading to a sustainable path of greater value.

With the following example, we show that it is possible to construct an asymmetric path with higher average value for $n > 2$.

Example 6. Let $n = 3$, $k = 3$, $\beta = 0.2$, $c = 1$, and $\delta = 0.2$. This results in $\alpha_1 = 0.3707$, $\alpha_2 = 0.3307$, $\alpha_3 = 0.2987$. The optimal symmetric price path has $p_1 = 1$, $p_2 \approx 0.6082$, $p_3 \approx 0.1135$, and value $v_1 \approx 0.3784$.

On the optimal asymmetric price path, all firms set $p_1 = 1$ and $p_3 \approx 0.113$, while firms set different prices in period 2: $p_2^1 = 0.6733$ and $p_2^2 = p_2^3 = 0.6033$. This path has value for each player of $v_1^1 \approx 0.3781$ and $v_1^2 = v_1^3 \approx 0.3802$, yielding average value $v_1 = (v_1^1 + v_1^2 + v_1^3)/3 \approx 0.3795$.

The average value of the asymmetric path is higher than the symmetric path. Relative to the symmetric path, one of the firms has a higher price in period 2 and a lower value of the path, while the other two firms set a lower price in period 2 and receive a higher value. If a cartel is contemplating whether to adopt the symmetric or asymmetric path, transfers may be required if the asymmetric path is chosen. □

As in our main model, the price path is determined by undercutting constraints. When $n = 2$, in periods benefitting from obfuscation, each undercutting constraint involves a market share of α in both the symmetric and asymmetric programs. However, for $n = 3$, this is no longer the case. In the asymmetric program, some undercutting constraints involve market shares of α_1, while others involve α_2. With $\alpha_2 < \alpha_1$, this permits a relaxation of the incentive constraints and higher attendent prices in the asymmetric case, leading to a higher average value.

4.2.2 Mixed strategies

The following example illustrates the potential benefit of (symmetric) mixed strategies.

Example 7. Let $n = 2$, $k = 3$, $\beta = 0.375$, $c = 1$, and $\delta = 0.2$. This results in $\alpha = 0.5703$, $\gamma = 0.4297$. The optimal pure strategy path involves price $p_1^p = 1$, $p_2^p \approx 0.6539$ and $p_3^p \approx 0.2298$ and value $v_1^p \approx 0.5746$, where the p superscript makes explicit the restriction to pure strategies.

Now consider the use of mixed strategies. In order to provide the certain protection of obfuscation, p_3 must be fixed. We can also show that $p_1 = 1$ is optimal. Thus, the opportunity for mixing is confined to the intermediate price p_2. Let $G(p_2)$ describe the cdf of p_2. Every price in the support of G must be equally profitable. On the optimal path, $p_3 \approx 0.238$ and G has support $[\gamma/\alpha, 1]$ with $G(p_2) = \frac{\alpha p_2 - \gamma}{p_2(\alpha-\gamma)}$. Note also that $E[p_2] = \int_{\gamma/\alpha}^{1} \frac{\gamma}{p(\alpha-\gamma)} dp = \frac{\gamma}{\alpha-\gamma} \ln \frac{\alpha}{\gamma} \approx 0.865$. This results in a higher value $v_1 \approx 0.596$. We illustrate the distribution of p_2 and the other prices on the pure strategy and mixed strategy paths in Figure 9.

Because firms mix on the continuous support $[\gamma/\alpha, 1]$, undercutting in the range $(p_3, \gamma/\alpha)$ offers no discrete jump in market share. Therefore, the marginal undercutting constraints no longer apply in period 2. The expected value of p_2 is higher under the mixed strategy path. This relaxes all incentive constraints and allows a higher value of p_3 relative to the pure strategy path. □

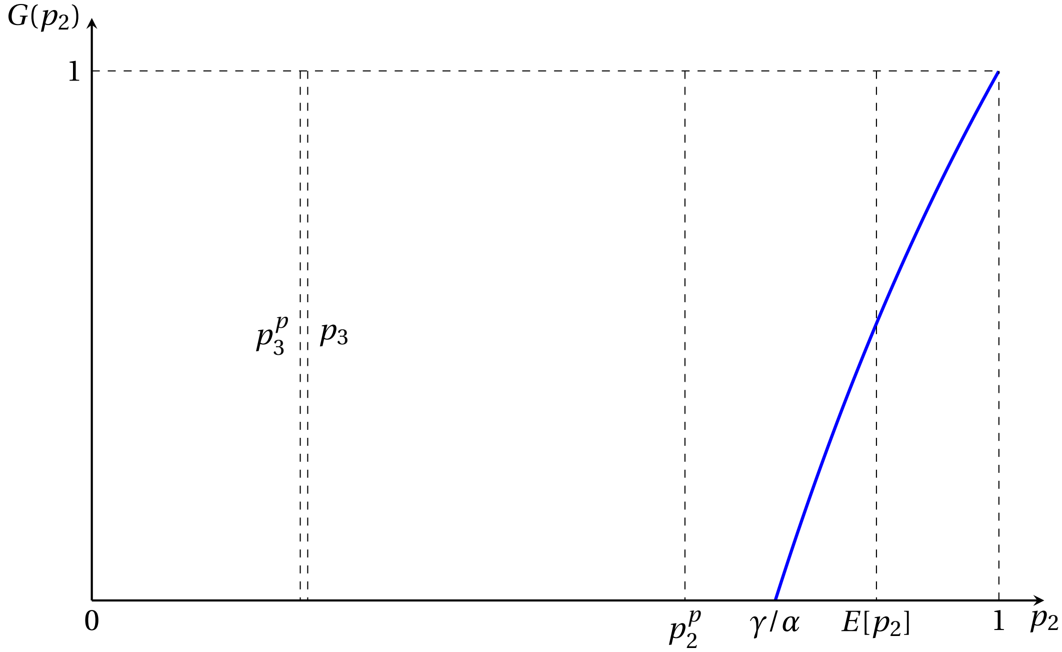
4.3 Forward-looking consumers

In our model, consumers purchase every period. An alternative possibility is that some consumers learn and exploit the intertemporal variation in prices by timing their purchases. In this section, we consider an example with this feature.

Suppose that in each period, a unit measure of consumers enters the market. A fraction $1-\theta$ of these consumers behave as before. They are impatient, and purchase in the current period. The remaining fraction θ are patient and forward-looking consumers. In particular, these consumers have discount factor $\delta = 1$ and are able to anticipate perfectly the dynamic path of prices. On the equilibrium path, they stay in the market until they observe the lowest price in a cycle, in period k. Consequently, in period s of a cycle, there will be a measure $1-\theta$ of impatient consumers and a measure $s\theta$ of forward-looking consumers. The impact on cartel policies is then summarised by Proposition 5.

Proposition 5. *Suppose $\delta < (n-1)/n$. Let $\{p_s^0\}_{s=1}^k$ solve the program (12) to (15), and let $\{v_s^0\}_{s=1}^k$ be the associated cartel continuation values. Then the solution to the corresponding problem*

Figure 9: Mixed strategies



with fraction θ *of forward looking consumers involves optimal cartel prices of*

$$p_s = p_s^0,\ s = 1, \ldots, k-1, \qquad\qquad p_k = \left(1 + \frac{k\theta}{1-\theta}\right)^{-1} p_k^0.$$

The associated continuation values are

$$v_s = (1-\theta)\, v_s^0,\ s = 1, \ldots, k.$$

Forward-looking consumers wait until the low point in the cycle to purchase, leading to a build-up of consumers in period k. For a given price p_k, this raises the profitability of deviation in period k, placing greater strain on the incentive compatibility constraints of the cartel. The remedy is to have a deeper sale that exactly offsets the increased demand in period k. There is no other impact on the optimal price path, and there is no impact on cartel sustainability. Notice also that forward-looking consumers are of no benefit to the cartel. Their contribution to market demand is exactly offset by the need for a deeper sale. However forward-looking consumers do offer an external benefit to their impatient peers, leading to lower prices for those impatient consumers fortunate enough to enter the market in period k.

The same argument extends to more general forms of forward-looking consumer behaviour. If some consumers are patient and anticipate lower prices, demand will rise as prices fall through a cycle. To balance internal incentive constraints, the cartel simply needs to accelerate price cutting during the cycle.

4.4 Pricing dynamics and market characteristics

In this section, we discuss the implications of our model for comparison across markets. The obfuscation properties of a price path are summarised by the salience relation $\phi(p, y)$ and the attentiveness parameter β. In markets requiring frequent purchases, consumers may develop a clearer understanding of the price distribution. Markets with highly visible prices may also foster consumer price awareness. We can capture such variation across markets in our model by adjusting β.

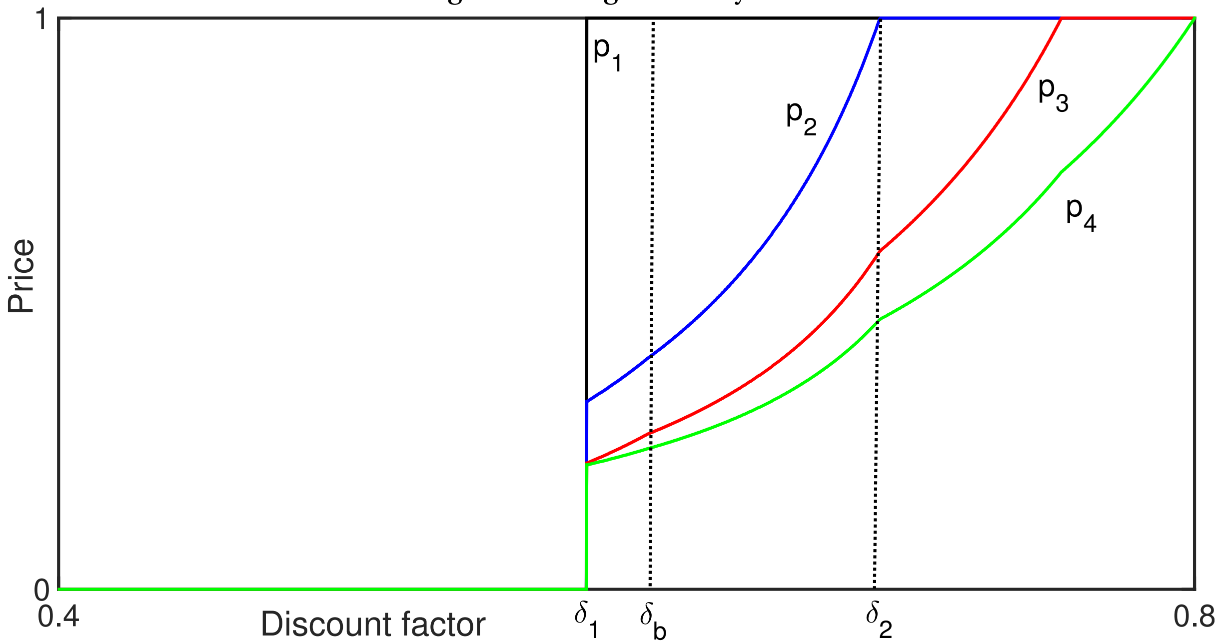
In a more general framework, the power to obfuscate may depend on other properties of the price path such as the complexity of the path or the number of distinct prices in a cycle. More complicated paths may achieve greater obfuscation. For example, pure sales paths may be more transparent to consumers than distinct cycle paths. This opens up the possibility that firms may be able to overcome the attentiveness of consumers through greater efforts at obfuscation. We could see more intricate price paths in market settings with highly visible prices and frequent purchases.

We illustrate this point with two industry examples. Asymmetric cycles that resemble distinct cycle paths are a common feature of retail petrol markets, while sales are endemic in the market for retail groceries.[23] Let us discuss first the market for petrol. In this market, consumers purchase frequently and prices are prominently displayed on billboards that are easily observed while driving. One might suspect that these conditions are unsuitable for obfuscation. Nevertheless, coordinated asymmetric cycles are common. Consider two potential explanations that are consistent with our model. First, the attentiveness parameter may depend on the complexity of the price path. If β decreases in the length or complexity of price cycles, then distinct cycle paths may be easier to sustain than sales paths. Such paths are quite opaque to consumers while also minimising the contemporaneous benefits of consumer search. Second, if attentiveness varies over the distribution of prices as in (23), then distinct cycle paths may provide greater protection against deviation relative to pure sales paths. We illustrate with the following example.

Example 8. Let $n = 5$, $k = 4$, and $(\beta^{(1)}, \beta^{(2)}, \beta^{(3)}) = (0.25, 0.5, 0.75)$. Figure 10 depicts the equilibrium price path. For $\delta \geq \delta_b$, the solution is described by Proposition 3 and the prices p_2, p_3, and p_4 are determined by binding marginal undercutting constraints. When $\delta \leq \delta_b$, the one-step undercutting constraint $v_2 \geq \alpha_1^{(3,2)} p_3$ binds and determines p_3 instead of the

[23] Explaining the source of pricing dynamics in these markets is beyond the scope of the current paper, and we do not claim to do so here. For a discussion of coordination of asymmetric cycles in retail petrol markets, see Byrne and de Roos (2015). On the relationship between retail petrol price dynamics and cost and demand variation, see, for example, Noel (2007a).

Figure 10: Edgeworth cycles



marginal undercutting constraint $v_3 \geq \alpha_1^{(3)} p_3$. The marginal undercutting constraint $v_2 \geq \alpha_1^{(2)} p_2$ also binds and determines the value of p_2. As a result, p_2 and p_3 are directly proportional, $p_2/p_3 = \alpha_1^{(3,2)}/\alpha_1^{(2)}$.

For $\delta \in [\delta_1, \delta_2)$, the optimal path is a distinct cycle path.[24] Comparing this solution to that of Example 1 in Figure 3, we see that distinct cycle paths account for a much greater range of discount factors. This is because distinct cycle paths provide greater obfuscation if attentiveness varies according to (23). In a pure sales path, $\phi(p_s, \{y_i\}_{i=1}^4) = \beta^{(3)}$ for $s = 1,2,3$. In a distinct cycle path, $\phi(p_s, \{y_i\}_{i=1}^4) = \beta^{(s)}$ for $s = 1,2,3$. □

The market for retail groceries is markedly different. When consumers decide where to purchase groceries, the price comparison exercise is a complicated one. Many products are included in the grocery bundle, and the price of each product may not be surveyed on each shopping excursion. In addition, only a single retailer is typically observed at a time. In these circumstances, only modest efforts may be required from retailers to obfuscate the consumer price comparison exercise. Occasional sales for a subset of the product space may be sufficient to hinder consumers in these efforts. Moreover, such sales may not need to be synchronised between firms.

[24] Notice that for $\delta < \delta_b$, $\delta_1 \approx 0.59$ is not determined by marginal undercutting constraints as in (28).

Appendices

A Proofs

Proof of Lemma 1

Proof. If all prices are distinct and $p_{(1)} > y$, a consumer will purchase at a price $p_{(r)}$ if the $r-1$ lower prices are not salient and either $p_{(r)}$ is salient or no prices are salient. If no prices are salient, the consumer searches until she observes a price no higher than z. Thus, (1) implies that α_r is determined by (6).

Next, consider the cases in (5). If $p^j = p_{(1)} = p_{(l)} \leq y$, then by (1), the l lowest prices are salient and $s^j = 1/l$. If $p^j > z$, consumers will continue searching after observing p^j. If $p_{(1)} \leq y$ and $p^j > p_{(1)}$, then consumers recall the price $p_{(1)}$ and therefore purchase from a store setting price $p_{(1)}$ without further search. Otherwise, we must have $p_{(1)} > y$. If $p^j = p_{(q)} = p_{(q+l-1)}$, then this price is shared between l competitors and $s^j = \sum_{r=q}^{q+l-1} \alpha_r / l$. □

Proof of Lemma 2

Proof. Suppose the k-period cycle $\sigma^{(k)}$ prescribes the constant price path $p_s = p \in (0,1]$, $s = 1,\ldots,k$. Then $y = p$. A deviation involving a higher price attracts a market share of 0 and is unprofitable. Deviation by undercutting yields a market share of 1. The strategy $\sigma^{(k)}$ gives value $p/(n(1-\delta))$, while deviation by marginally undercutting yields payoff approaching p. This deviation is unprofitable if and only if $\delta \geq (n-1)/n$. □

Proof of Lemma 3

Proof. It follows directly from (1) and (2) that $\sigma^{(k)}$ specifies an optimal penal code of zero value. A standard application of the one-shot deviation principle is also possible. See, for example, Fudenberg and Tirole (1991). We restrict attention to one-shot deviations below. We first show that the prevention of marginal undercutting leads to the constraints in (13). Suppose following history h^t, $\sigma^{(k)}$ calls on players to choose price p_s with $p_s \geq p_k$. Then the continuation value induced by $\sigma^{(k)}$ following h^t is v_s. If $p_s > p_k$, a marginal undercut of rival prices yields market share α_1. The restriction $v_s \geq \alpha\pi_s$ is therefore sufficient to deter this deviation. If instead $p_s = p_k$, undercutting leads to market share 1 rather than α_1, and this case is dealt with in the constraints (14). Similarly, marginal undercutting of the price associated with p_k leads to a market share of 1, leading to the last constraint in (13).

Next, we show that no other deviations involving undercutting the current price vector are possible if the constraints in (14) are satisfied. Suppose $\sigma^{(k)}$ calls on p_s to be played in the current period with $p_s > p_k$. Any deviation $p' \in [p_k, p_s)$ will yield market share α_1, making a marginal undercut the most profitable deviation. Alternatively, undercutting below p_k will yield market share of 1. This deviation is not profitable if $v_s \geq p_k$, leading to the constraints in (14).

Finally, consider a deviation $p' > p_s$ in period s. If $p' > 1$, consumers will not purchase, and if $p' > z_s$, consumers will continue searching. The optimal upward deviation is therefore $p' = \min\{1, z_s\}$. This deviation is never profitable if $v_s \geq \alpha_n \min\{1, z_s\}$. The constraints in (15) thus ensure that there is no incentive to raise prices. □

Proof of Lemma 4

Proof. 1) Initially, disregard the constraints in (14). Once we have described the equilibrium, we show that in equilibrium these constraints are satisfied. Number the constraints in (13) $1, \ldots, k$, where constraint s involves v_s.

2) As a preliminary step let us prove that the multi-price equilibrium exists only if $\delta > \frac{\alpha_1 n - 1}{\alpha_1 n}$. Add up all constraints in (13) to derive

$$((p_1 + p_2 + \cdots + p_k)(1 + \delta + \cdots + \delta^{k-1}))/(1 - \delta^k) \geq \alpha_1 n(p_1 + p_2 + \cdots + p_k) + (1 - \alpha_1) n p_k.$$

Simplify the above expression

$$(p_1 + p_2 + \cdots + p_k)\,(1/(1 - \delta) - \alpha_1 n) \geq (1 - \alpha_1) n p_k.$$

Given that prices are positive it follows that $\frac{1}{1-\delta} > \alpha_1 n$, which means that $\delta > \frac{\alpha_1 n - 1}{\alpha_1 n}$.

3) Show that for each $s = 1, \ldots, k$, constraint s is either binding or $p_s = 1$. Consider constraint s and suppose otherwise that the constraint is not binding and that $p_s < 1$. By increasing p_s until either $p_s = 1$ or constraint s is binding, the objective function v is increased and all the constraints in (13) are satisfied, leading to a contradiction.

4) Observe that constraint k must be binding. Suppose otherwise that $p_k = 1$. Then (11) and the last constraint imply $v_k = \delta v_1 + 1/n \geq 1$. With $\delta < \frac{n-1}{n}$, this means that $v \geq v_1 > 1$. Notice that setting price $p_s = 1$ for all s yields value $\frac{1}{n(1-\delta)} < 1$, leading to a contradiction. Consequently, $p_k < 1$ and the last constraint is satisfied with equality, $v_k = p_k$. Using (11), note that $p_k = \frac{\delta n}{n-1} v_1$.

5) Next, we show that for each $s = 2, \ldots, k$, if $p_s = 1$ then $p_{s-1} = 1$. If $p_s = 1$ then constraint s becomes $v_s \geq \alpha_1$. Constraint $s-1$ stipulates $v_{s-1} \geq \alpha_1 p_{s-1}$. Using equality $v_{s-1} = \delta v_s + p_{s-1}/n$

from (11), transform this to $v_s \geq \frac{\alpha_1 n-1}{n\delta} p_{s-1}$. With $\delta > \frac{\alpha_1 n-1}{\alpha_1 n}$, this implies that $\alpha_1 n > \frac{\alpha_1 n-1}{\delta}$. Consequently $p_{s-1} = 1$ satisfies constraint $s-1$.
6) We now show that $p_{k-1} = \min\left\{1, \frac{\delta n}{\alpha_1 n-1} p_k\right\}$. Using (11) to transform constraint $k-1$, we obtain $\delta v_k + p_{k-1}/n \geq \alpha_1 p_{k-1}$. Observing that constraint k is binding and rearranging, we obtain $p_{k-1} \leq \frac{\delta n}{\alpha_1 n-1} p_k$. Applying point 3) above yields our desired result.
7) Similarly, we show that $p_s = \min\left\{1, \frac{\delta\alpha_1 n}{\alpha_1 n-1} p_{s+1}\right\}$, $s = 2, \ldots, k-2$. First, note that if constraint $s+1$ is not binding then $p_s = 1$. Alternatively, suppose constraint $s+1$ is binding. We can use (11) to transform constraint s to obtain $\delta v_{s+1} + p_s/n \geq \alpha_1 p_s$. Using the fact that constraint $s+1$ is binding, we obtain $\delta\alpha_1 p_{s+1} + p_s/n \geq \alpha_1 p_s$ or $p_s \leq \frac{\delta\alpha_1 n}{\alpha_1 n-1} p_{s+1}$. Applying point 3) above yields our desired result.
8) Let us prove that $v = v_1$. Note that points 6) to 7) and the fact that $\delta > \frac{\alpha_1 n-1}{\alpha_1 n}$ imply $p_1 \geq p_2 \geq \cdots \geq p_{k-1} > p_k$; this result follows immediately.
9) We now show that $p_1 = 1$. Notice that all constraints are homogeneous of degree one with respect to prices. We first prove by contradiction that $\max_i p_i = 1$. With a choke price of 1, $p_i \leq 1\ \forall\ i$. Suppose that $\max_i p_i < 1$. Introduce the variables $p'_s = p_s/\max_i p_i\ \forall\ s \leq k$. Because of first degree homogeneity, our transformed variables must satisfy the constraints. Given that $p'_s > p_s\ \forall\ i$, this means that $v' > v$. Consequently, there is a contradiction and $\max_i p_i = 1$. Monotonicity of p_s in s then ensures that $p_1 = 1$.
10) Next, we show that in equilibrium the constraints in (14) are satisfied. Consider two separate cases: $p_{k-1} < 1$ and $p_{k-1} = 1$. If $p_{k-1} < 1$ then from point 6) $p_{k-1} = \frac{\delta n}{\alpha_1 n-1} p_k$. This implies that $\alpha_1 p_{k-1} > p_k$. Combined with the fact that $p_1 \geq p_2 \geq \cdots \geq p_{k-1}$ and the constraints in (13), this proves that the constraints in (14) hold. If $p_{k-1} = 1$ then using 5), $p_1 = p_2 = \cdots = p_{k-1} = 1$. This means $v_k < v_s$ for any s. Combined with the constraint k in (13), this implies that the constraints in (14) hold.
11) Let us show that $\forall\ \delta$, the optimal sequence $\{p_i\}_{i=1}^k$, if it exists, must be unique. Assume the opposite that there are two sequences. v_1 must be the same for both sequences, otherwise the one with the lower v_1 is dominated. This will uniquely determine the value of p_k and then, recursively, $p_{k-1}, \ldots, p_1$. The optimal sequence is therefore unique.
12) Next, derive δ_1. Consider the situation when all constraints are binding. From constraint $k-1$ it follows that $v_{k-1} = \alpha_1 p_{k-1}$. Using (11) it follows that $v_k = \frac{\alpha_1 n-1}{\delta\alpha_1 n} v_{k-1}$. Continuing this process results in

$$v_k = \left(\frac{\alpha_1 n-1}{\delta\alpha_1 n}\right)^{k-i} v_i, \qquad i = 1, \ldots, k-1. \tag{30}$$

In particular, $v_k = \left(\frac{\alpha_1 n-1}{\delta\alpha_1 n}\right)^{k-1} v_1$. Then, using (11) and $v_k = p_k$ from constraint k, we obtain $\delta_1 = \left(\frac{\alpha_1 n-1}{\alpha_1 n}\right)^{\frac{k-1}{k}} \left(\frac{n-1}{n}\right)^{\frac{1}{k}}$.

13) Next, note that $v_1(\delta)$ is strictly increasing in δ, where $v_1(\delta)$ is the value of v_1 associated with the optimal price sequence for discount factor δ. Consider any two discount factors δ and δ' with $\delta' > \delta$. Let $p^k = \{p_i\}_{i=1}^k$ be the optimal price sequence associated with δ and p'^k be the corresponding sequence for δ'. Then, abusing notation slightly, $v_1(\delta', p'^k) > v_1(\delta', p^k) > v_1(\delta, p^k)$, as required.
14) We now show that a multi-price equilibrium exists if and only if $\delta_1 \leq \delta < \frac{n-1}{n}$. Recall that when $\delta = \delta_1$, constraint 1 in (13) is binding and $v_1 = \alpha_1$. Because v_1 is strictly increasing in δ, if $\delta < \delta_1$ then $v_1 < \alpha_1$, leading to violation of constraint 1. Similarly, if $\delta > \delta_1$ then $v_1 > \alpha_1$ and all constraints are satisfied.
15) Finally, we characterise the optimal path in terms of values rather than prices. Using constraint k and (11), we obtain $v_k = \frac{n\delta}{n-1} v_1$. Next, consider the constraints for each period s for $s < k$. Noting that $p_s \leq 1$ and using (11) yields $v_s \leq 1/n + \delta v_{s+1}$. Using constraint s and (11) leads to $v_s \leq \frac{\delta \alpha_1 n}{\alpha_1 n - 1} v_{s+1}$. Employing the complementary slackness conditions of 3) then yields $v_s = \min\{1/n + \delta v_{s+1}, \frac{\delta \alpha_1 n}{\alpha_1 n - 1} v_{s+1}\}$. □

Proof of Lemma 5

Proof. At the knot discount factor δ_s, $p_s = 1$, and constraint s is binding so that $v_s = \alpha_1$. Using equation (39), we can then show $v_1 = \frac{v_k(n-1)}{n\delta} = \left(\frac{\alpha_1 n - 1}{\alpha_1 n}\right)^{k-s} \frac{n-1}{n} \frac{\alpha_1}{\delta^{k-s+1}}$. On the other hand, recursively employing (11), v_1 can be represented as $v_1 = \delta v_2 + 1/n = \cdots = \delta^{s-1} v_s + \delta^{s-2}/n + \cdots + 1/n = \alpha_1 \delta^{s-1} + \delta^{s-2}/n + \cdots + 1/n$. Combining both relationships gives

$$\delta^k + \frac{1}{\alpha_1 n} \sum_{i=1}^{s-1} \delta^{k-i} = \left(\frac{\alpha_1 n - 1}{\alpha_1 n}\right)^{k-s} \left(\frac{n-1}{n}\right).$$

The lemma therefore is proved. □

Proof of Lemma 6

Proof. First let $s = k-1$, that is consider the range $\delta \in [\delta_{k-1}, \delta_k)$ and recall that there is a sales path in this range. Using (11), note that $v_1 = \frac{(n-1)p_k}{n\delta}$. Also, applying the definition of v_1 directly gives $v_1 = \frac{1+\delta+\cdots+\delta^{k-2}+\delta^{k-1}p_k}{n(1-\delta^k)}$. Combining results in $p_k = \frac{\delta(1-\delta^{k-1})}{(n(1-\delta^k)-1)(1-\delta)}$ and $v_1 = \frac{(n-1)(1-\delta^{k-1})}{n(1-\delta)(n(1-\delta^k)-1)}$.

Now using a similar approach let us prove the statement for any $s = 1, \ldots, k-2$. That is, consider the range $\delta \in [\delta_s, \delta_{s+1})$. In this case condition $v_1 = \frac{(n-1)p_k}{n\delta}$ still holds, while the second condition transforms to $v_1 = \frac{1+\delta+\cdots+\delta^{s-1}+\delta^s p_{s+1}+\cdots+\delta^{k-1}p_k}{n(1-\delta^k)}$. Note that for $\delta \in [\delta_s, \delta_{s+1})$, the following condition holds: $p_k < p_{k-1} < \cdots < p_{s+1} < 1$. From Lemma 4 it then follows that

$p_{k-1} = \frac{\delta n}{\alpha_1 n-1} p_k$, $p_{k-2} = \frac{\delta n}{\alpha_1 n-1}\frac{\delta \alpha_1 n}{\alpha_1 n-1} p_k, \dots, p_{s+1} = \frac{\delta n}{\alpha_1 n-1}\left(\frac{\delta \alpha_1 n}{\alpha_1 n-1}\right)^{k-s-2} p_k$. Combining these conditions results in $p_k = \frac{\delta(1-\delta^s)}{(n(1-(\frac{\alpha_1 n}{\alpha_1 n-1})^{k-s-1}\delta^k)-1)(1-\delta)}$ and $v_1 = \frac{(n-1)(1-\delta^s)}{n(1-\delta)(n(1-(\frac{\alpha_1 n}{\alpha_1 n-1})^{k-s-1}\delta^k)-1)}$.

Continuity follows from Lemma 5. The denominator of p_k and v_1 are non-zero for $\delta \in [\delta_s, \delta_{s+1})$, establishing continuity in δ for this region. The lemma therefore is proved. □

Proof of Lemma 7

Proof. 1) Consider constraint s of (15), where $s = 1, \dots, k-1$. If $p_s \geq 1-c$, then (15) implies that $p_s \geq \alpha_n$. If $p_s < 1-c$, then (15) implies that $p_s \geq \alpha_n(p_s + c)$, and therefore $p_s \geq \alpha_n c/(1-\alpha_n)$. Recall that p_s is decreasing in s. Thus, (15) is equivalent to the implicit condition

$$p_{k-1} \geq \begin{cases} \alpha_n/\alpha_1 & \text{if } p_{k-1} \geq 1-c, \\ \frac{\alpha_n c}{(1-\alpha_n)\alpha_1} & \text{if } p_{k-1} < 1-c. \end{cases} \tag{31}$$

2) Suppose that $\alpha_n \geq 1-c$, and observe that this is equivalent to the condition $\alpha_n c/(1-\alpha_n) \geq \alpha_n$. If $p_{k-1} \geq \alpha_n/\alpha_1$, then $p_{k-1} \geq 1-c$ and (31) is satisfied. Next, consider the case $p_{k-1} < \alpha_n/\alpha_1$. The condition $\alpha_n \geq 1-c$ implies that $\alpha_n c/(1-\alpha_n) \geq \alpha_n$ and therefore that $p_{k-1} < \alpha_n c/((1-\alpha_n)\alpha_1)$, and (31) is not satisfied. Therefore, when $\alpha_n \geq 1-c$, $p_{k-1} \geq \alpha_n/\alpha_1$ is equivalent to (31).

3) Suppose instead that $\alpha_n < 1-c$. This is equivalent to the condition $\alpha_n c/(1-\alpha_n) < \alpha_n$. Consider first $p_{k-1} \geq \alpha_n c/((1-\alpha_n)\alpha_1)$. If $p_{k-1} < 1-c$, then by 1) above, (31) is satisfied. If instead $p_{k-1} \geq 1-c$, then because $\alpha_n < 1-c$, $p_{k-1} \geq \alpha_n/\alpha_1$, and (31) again holds. Thus, if $p_{k-1} \geq \alpha_n c/((1-\alpha_n)\alpha_1)$, then (31) is satisfied. Next, examine the case $p_{k-1} < \alpha_n c/((1-\alpha_n)\alpha_1)$. This implies $\alpha_n/\alpha_1 > p_{k-1}$. This contradicts (31). Therefore, when $\alpha_n/\alpha_1 < 1-c$, $p_{k-1} \geq \alpha_n c/((1-\alpha_n)\alpha_1)$ is equivalent to (31).

4) Observe that 1), 2) and 3) imply that the relenting constraints (15) are satisfied iff $p_{k-1} \geq \gamma$, where γ is defined in (19). □

Proof of Lemma 8

Proof. From Lemma 4, $p_k = \delta v_1 n/(n-1)$ and $p_{k-1} \leq \frac{\delta n}{\alpha_1 n-1} p_k$. The lowest value of v_1 occurs when all constraints are binding and $\delta = \delta_1$. Therefore, $v_1 = \alpha_1$ and $p_{k-1} = \delta_1^2 n^2 \alpha_1/((n-1)(\alpha_1 n-1))$. Hence, $\delta_1^2 n^2 \alpha_1 \geq \gamma(n-1)(\alpha_1 n-1)$. Substituting for δ_1 and rearranging, we obtain

$$\gamma \leq \left(\frac{\alpha_1 n-1}{\alpha_1(n-1)}\right)^{\frac{k-2}{k}}. \tag{32}$$

Equation (19) implies that $\gamma > 0$ if $\beta = 0$ and $c > 0$. Therefore, (32) is violated when $\beta = 0$. Next, observe that when $\beta = 1$, $\gamma = 0$ and the right hand side of (32) is positive, and therefore (32) is satisfied. Finally, note that for $c > 0$, $\gamma'(\beta) < 0$, and the right hand side of (32) is strictly increasing in β. Therefore, there exists β_1 such that for any $\beta \geq \beta_1$, the prices $\{p_s\}_{s=1}^{k}$ also satisfy the constraints in (15), and for $\beta < \beta_1$, they do not. This establishes part i).

Observe that γ is independent of δ. Because v_1 is continuous and strictly increasing in δ, $p_k = \frac{\delta v_1 n}{n-1}$ and $p_{k-1} = \min\left\{1, \frac{\delta n}{\alpha_1 n-1} p_k\right\}$, p_{k-1} is a strictly increasing continuous function of δ. If $\delta = \frac{n-1}{n}$, a fixed price can be supported in equilibrium and therefore $p_{k-1} = 1 > \gamma$. If $\delta = 0$, then no collusive equilibrium exists and we must have $p_{k-1} = 0 < \gamma$. Therefore, there is a unique $\hat{\hat{\delta}}$ such that $\gamma(\beta) = p_{k-1}(\beta, \hat{\hat{\delta}})$. An equilibrium exists if and only if $\delta \geq \hat{\hat{\delta}}$. This establishes part ii).

For any β and n, (19) implies that as $\overline{c}$ approaches 0, $\gamma(n, \beta, c)$ converges to 0 for any $c < \overline{c}$. From Lemma 4, recall $p_k = \delta n v_1/(n-1) > 0$ and $p_{k-1} = \min\left\{1, \frac{\delta n}{\alpha_1 n-1} p_k\right\}$. Hence, for any β, there exists $\overline{c}$ such that for any $c < \overline{c}$, $\gamma(n, \beta, c) < p_{k-1}$, and the relenting constraints in (15) do not bind. This establishes part iii). □

Proof of Lemma 9

Proof. The knot discount factors are characterized by $v_s = \alpha_1$ for $s = 2, \ldots, k$, and equation (17). We can represent $v_1 = \delta v_2 + 1/n = \cdots = \delta^{s-1} v_s + \delta^{s-2}/n + \cdots + 1/n = \delta^{s-1}\alpha_1 + \delta^{s-2}/n + \cdots + 1/n$.

Substitute v_1 into equation (17) to derive

$$\delta v_1 = \left(\frac{\alpha_1 n - 1}{\alpha_1 n}\right)^{k-s} \left(\frac{n-1}{n}\right) \alpha_1 \delta^{s-k}. \tag{33}$$

Using Lemma 4 and substituting δv_1 from equation (33), we derive

$$p_{k-1} = \left(\frac{\alpha_1 n - 1}{\alpha_1 n}\right)^{k-s-1} \delta^{s+1-k}. \tag{34}$$

The s-th zone exists whenever $p_{k-1} \geq \alpha_n/\alpha_1$ is feasible. The critical value of δ is then $\delta = \frac{\alpha_1 n-1}{\alpha_1 n}\left(\frac{\alpha_1}{\alpha_n}\right)^{\frac{1}{k-s-1}}$. Substitute this value in equation (17) and simplify to derive

$$\left(\frac{\alpha_1 n - 1}{\alpha_1 n}\right)^{s} \left(\frac{\alpha_1}{\alpha_n}\right)^{\frac{k}{k-s-1}} + \frac{1}{\alpha_1 n} \sum_{i=1}^{s-1} \left(\frac{\alpha_1 n - 1}{\alpha_1 n}\right)^{s-i} \left(\frac{\alpha_1}{\alpha_n}\right)^{\frac{k-i}{k-s-1}} = \frac{n-1}{n}. \tag{35}$$

Define the terms

$$X = \frac{\alpha_1 n - 1}{\alpha_1 n}, \qquad Y = \frac{\alpha_1}{\alpha_n},$$

and note that $0 < X < 1$, $Y > 1$, and both X and Y are increasing functions of β. Use X and Y to rewrite equation (35),

$$X^s Y^{\frac{k}{k-s-1}} + (1-X)\left(X^{s-1} Y^{\frac{k-1}{k-s-1}} + \cdots + X Y^{\frac{k-s+1}{k-s-1}}\right) = \frac{n-1}{n}.$$

With some manipulation, we obtain

$$X^s\left(Y^{\frac{s-1}{k-s-1}} - Y^{\frac{s-2}{k-s-1}}\right) + X^{s-1}\left(Y^{\frac{s-2}{k-s-1}} - Y^{\frac{s-3}{k-s-1}}\right) + \cdots + X = \frac{n-1}{n} Y^{-\frac{k-s+1}{k-s-1}}.$$

The left hand side of this expression is increasing in β while the right hand side is decreasing in β. Also, note that if $\beta = 0$, then $X = 0$ which implies that the left hand side is equal to zero. Alternatively, if $\beta = 1$, then $\alpha_n = 0$ and $\alpha_1 = 1$ and the left hand side becomes infinite. Therefore, there is a unique solution $\beta = \beta_s$. Increasing β will increase p_{k-1} (see equation (34)), decrease α_n, and make the constraint $p_{k-1} \geq \alpha_n/\alpha_1$ easier to satisfy. Consequently, it is proved that for $\beta > \beta_s$ the s-th zone exists. The lemma therefore is proved. □

Proof of Proposition 1

Proof. The result follows directly from Lemmas 2 to 9. □

Proof of Proposition 2

Proof. We use the notation $x = \{p_i\}_{i=1}^{\infty}$ to denote a price path, and $x_r^s = \{p_r, \ldots, p_s\}$ to denote a subpath. We first consider the optimal path abstracting from initial conditions. In the final step, we impose the initial condition $y_o = \infty$.

1) Consider any consecutive supporting prices p_r and p_s with $r < s$. Every price p_i for $r < i < s$ is supported but not supporting. Therefore, p_i must satisfy $v_i \geq p_r$, and either $p_i = 1$ or $v_i = \alpha_1 p_i$. Based on these conditions, we established in Lemma 4 that: for $r+1 < i < s$, if $p_i = 1$, then $p_{i-1} = 1$; and for $r < i \leq s$, p_i and v_i are decreasing in i.

2) We now show that no supporting price is reliant on another price. Suppose otherwise that the supporting price p_r is reliant, $v_r < p_r$. If $s > r+1$, then $v_{s-1} \geq p_r$, which implies $v_{s-1} > v_r$. We can therefore increase the value of the path without violating any constraints by excising the subpath x_r^{s-2}. If instead $s = r+1$ then, because p_r is supporting, we must have $p_r < p_s$. Hence, $v_s \geq p_r$. This implies $v_s > v_r$, and therefore excising p_r will increase the value of the path without violating any constraints.

3) Next, we show that $v_r = p_r$ for every supporting price p_r. In an optimal path, the supporting price p_r is maximised subject to the constraints $v_r \geq p_r$, $v_{s-1} \geq p_r$, and $v_s \geq p_r$. At least one of these constraints must be satisfied with equality. The first constraint follows

because v_r is not reliant. Recall that p_{s-1} is the lowest price among the supported prices between p_r and p_s. The second constraint prevents undercutting below p_r in period $s-1$. This constraint applies only if $s > r+1$. The third constraint prevents undercutting below p_r for period s in the event that $p_r < p_s$. Suppose first that $v_s = p_r$ and $p_r < p_s$. However, p_s is not reliant, which implies $v_s \geq p_s$, leading to a contradiction. Next, suppose $v_{s-1} = p_r$. This implies $v_s < p_r$, and thus $v_r > v_s$. Therefore, if x_r^{s-1} is repeated forever from period s, no constraints will be violated and the value of the path will be increased. Hence, the first constraint must bind, as contended.

4) Given the consecutive supporting prices, p_r, p_s, and p_t, focus attention on the associated subpaths x_r^{s-1} and x_s^{t-1}. We show that these subpaths are identical. Suppose that $p_s < p_r$. This implies $v_s < v_r$. Because no price outside the subpath x_s^{t-1} is reliant on p_s, the subpath x_s^{t-1} can be replaced by the subpath x_r^{s-1} without violating any incentive constraints. This would increase the value of the path, leading to a contradiction. Alternatively, suppose $p_r < p_s$, and hence $v_r < v_s$. No price outside the subpath x_r^{s-1} is reliant on p_r, and therefore the subpath x_r^{s-1} can be replaced by the subpath x_s^{t-1} without violating any constraints. This again increases the value of the path, leading to a contradiction. Hence, $p_r = p_s$. Finally, recall that the complementary slackness conditions described in 1) (and in more detail in Lemma 4) completely determine the remaining prices in each subpath, ensuring each subpath is identical. Extending the argument to all consecutive subpaths, it follows that all subpaths are identical.

5) Note that with memory m, for any consecutive supporting prices p_r and p_s, the subpath x_r^{s-1} is no longer than $m+1$ periods. Consider a repeated k period subpath, and let p_r be the supporting price. Up to m periods of the path can be supported by p_r, while p_r requires no support. Therefore, $k \leq m+1$.

6) Next, observe that the k period path with $k = m+1$ satisfies all the constraints of Lemma 3. For $s = 1, \ldots, m$, the price p_{m+1} is in consumer memory and the constraints are identical. In period $m+1$, the price p_{m+1} is no longer in memory, but this does not impact on the constraints. It is then a corollary of Proposition 1 that the optimal $m+1$ path with memory m is sustainable if and only if $\delta \geq \delta^*$.

7) We now show that the optimal $m+1$ period path dominates all shorter paths. Suppose instead that the path $\{\tilde{p}_s\}_{s=1}^k$ is optimal, with $k < m+1$. Prepend $p'_0 = 1$ to form the path $\{p'_s\}_{s=0}^k$, where $p'_s = \tilde{p}_s$ for $s = 1, \ldots, k$. Because $\tilde{p}_s \leq 1$ for $s = 1, \ldots, k-1$, and $\tilde{p}_k < 1$, the value of the new path is greater.

Next, notice that $v'_s > \tilde{v}_s$ for $s = 1, \ldots, k$. Hence, if the incentive constraints were satisfied for the original path, then they must also be satisfied in periods $s = 1, \ldots, k$ for the new path. Also observe that $p'_0 = p'_1 = 1$ and $v'_0 > v'_1$. Therefore, if the constraints in period 1 are satisfied, then the constraints in period 0 must also be satisfied. Thus, the $k+1$ period path is sustainable and of higher value than the k period path, leading to a contradiction. Therefore, the optimal path is of length $m+1$ periods.

8) It remains to specify how the path is initiated from the null history. Recall that $y_0 = \infty$. Thus, there is no finite price in memory in period 0. It follows that p_0 is the lowest price in memory at the commencement of period 1. That is, $y_1 = p_0$. Therefore p_0 is a supporting price. Combining with steps 1) - 7), it follows that $\tilde{\sigma}^{(m+1)}$ is both feasible and optimal. □

Proof of Proposition 3

Proof. (1) First, observe that $\{p_s\}_{s=1}^k$ and $\{v_s\}_{s=1}^k$ must satisfy the incentive constraints

$$v_s \geq \alpha_1^{(r,s)} p_r, \;\; r \geq s, s = 1, \ldots, k, \tag{36}$$

$$v_s \geq \alpha_n^{(r,s)} \min\{p_r, z_s\}, \;\; r < s, s = 1, \ldots, k. \tag{37}$$

The constraints in (36) deter deviation by undercutting the cartel price p_s, and the constraints in (37) deter relenting in period s. The derivation follows the argument of Lemma 3. Because of the more complex form of (23), undercutting and relenting to alternative cut-off prices must be considered. The relenting constraints are satisfied by assumption. We initially consider only the marginal undercutting constraints

$$v_s \geq \alpha_1^{(s)} p_s, \;\; s = 1, \ldots, k, \tag{38}$$

and suppose all other undercutting constraints are satisfied. In 15)-17) below, we show that these constraints do hold for the optimal path.

2) The condition $\delta \geq \max\{\delta_a, \delta_b\}$ implies that $\delta \geq \frac{\alpha_1^{(s-1)} n - 1}{\alpha_1^{(s)} n}$ for $s = 2, \ldots, k$.

3) We show that for each $s = 1, \ldots, k$, constraint s is either binding or $p_s = 1$. Consider constraint s and suppose otherwise that the constraint is not binding and that $p_s < 1$. By increasing p_s until either $p_s = 1$ or constraint s is binding, the objective function v is increased and all the constraints in (38) are satisfied, leading to a contradiction.

4) Observe that constraint k must be binding. Suppose otherwise that $p_k = 1$. Then (11) and the last constraint imply $v_k = \delta v_1 + 1/n \geq 1$. With $\delta < \frac{n-1}{n}$, this means that $v \geq v_1 > 1$. Notice that setting price $p_s = 1$ for all s yields value $\frac{1}{n(1-\delta)} < 1$, leading to a contradiction. Consequently, $p_k < 1$ and the last constraint is satisfied with equality, $v_k = p_k$. Using (11), note that $p_k = \frac{\delta n}{n-1} v_1$.

5) Next, we show that for each $s = 2, \ldots, k$, if $p_s = 1$ then $p_{s-1} = 1$. If $p_s = 1$ then constraint s becomes $v_s \geq \alpha_1^{(s)}$. Constraint $s-1$ stipulates $v_{s-1} \geq \alpha_1^{(s-1)} p_{s-1}$. Using equality $v_{s-1} =$

$\delta v_s + p_{s-1}/n$ from (11), transform this to $v_s \geq \frac{\alpha_1^{(s-1)}n-1}{n\delta} p_{s-1}$. With $\delta \geq \frac{\alpha_1^{(s-1)}n-1}{\alpha_1^{(s)}n}$, this implies that $\alpha_1^{(s)} \geq \frac{\alpha_1^{(s-1)}n-1}{n\delta}$. Consequently $p_{s-1} = 1$ satisfies constraint $s-1$.

6) We now show that $p_{k-1} = \min\left\{1, \frac{\delta n}{\alpha_1^{(k-1)}n-1} p_k\right\}$. Using (11) to transform constraint $k-1$, we obtain $\delta v_k + p_{k-1}/n \geq \alpha_1^{(k-1)} p_{k-1}$. Observing that constraint k is binding and rearranging, we obtain $p_{k-1} \leq \frac{\delta n}{\alpha_1^{(k-1)}n-1} p_k$. Applying point 3) above yields our desired result.

7) Similarly, we show that $p_s = \min\left\{1, \frac{\delta \alpha_1^{(s+1)}n}{\alpha_1^{(s)}n-1} p_{s+1}\right\}$, $s = 2, \ldots, k-2$. First, note that if constraint $s+1$ is not binding then $p_s = 1$. Alternatively, suppose constraint $s+1$ is binding. We can use (11) to transform constraint s to obtain $\delta v_{s+1} + p_s/n \geq \alpha_1^{(s)} p_s$. Using the fact that constraint $s+1$ is binding, we obtain $\delta \alpha_1^{(s+1)} p_{s+1} + p_s/n \geq \alpha_1^{(s)} p_s$ or $p_s \leq \frac{\delta \alpha_1^{(s+1)}n}{\alpha_1^{(s)}n-1} p_{s+1}$. Applying point 3) above yields our desired result.

8) Let us prove that $v = v_1$. Note that points 2), 6), and 7) imply $p_1 \geq p_2 \geq \cdots \geq p_{k-1} > p_k$; this result follows immediately.

9) We now show that $p_1 = 1$. Notice that all constraints are homogeneous of degree one with respect to prices. We first prove by contradiction that $\max_i p_i = 1$. With a choke price of 1, $p_i \leq 1 \ \forall\ i$. Suppose that $\max_i p_i < 1$. Introduce the variables $p'_s = p_s/\max_i p_i \ \forall\ s \leq k$. Because of first degree homogeneity, our transformed variables must satisfy the constraints. Given that $p'_s > p_s \ \forall\ i$, this means that $v' > v$. Consequently, there is a contradiction and $\max_i p_i = 1$. Monotonicity of p_s in s then ensures that $p_1 = 1$.

10) Let us show that $\forall\ \delta$, the optimal sequence $\{p_i\}_{i=1}^k$, if it exists, must be unique. Assume the opposite that there are two sequences. v_1 must be the same for both sequences, otherwise the one with the higher v_1 is chosen. This will uniquely determine the value of p_k and then, recursively, $p_{k-1}, \ldots, p_1$. The optimal sequence is therefore unique.

11) Next, derive δ_1. Consider the situation when all constraints are binding. From constraint $k-1$ it follows that $v_{k-1} = \alpha_1^{(k-1)} p_{k-1}$. Using (11) it follows that $v_k = \frac{\alpha_1^{(k-1)}n-1}{\delta \alpha_1^{(k-1)}n} v_{k-1}$. Continuing this process results in

$$v_k = \prod_{s=1}^{k-i} \left(\frac{\alpha_1^{(k-s)}n-1}{\delta \alpha_1^{(k-s)}n} \right) v_i, \qquad i = 1, \ldots, k-1. \tag{39}$$

In particular, $v_k = \prod_{s=1}^{k-1} \left(\frac{\alpha_1^{(k-s)}n-1}{\delta \alpha_1^{(k-s)}n} \right) v_1$. Then, using (11) and $v_k = p_k$ from constraint k, we obtain $\delta_1 = \prod_{s=1}^{k} \left(\frac{\alpha_1^{(s)}n-1}{\alpha_1^{(s)}n} \right)^{\frac{1}{k}}$.

12) Next, note that $v_1(\delta)$ is strictly increasing in δ, where $v_1(\delta)$ is the value of v_1 associated with the optimal price sequence for discount factor δ. Consider any two discount factors

δ and δ' with $\delta' > \delta$. Let $p^k = \{p_i\}_{i=1}^k$ be the optimal price sequence associated with δ and p'^k be the corresponding sequence for δ'. Then, abusing notation slightly, $v_1(\delta', p'^k) > v_1(\delta', p^k) > v_1(\delta, p^k)$, as required.

13) We now show that a multi-price equilibrium exists if and only if $\delta_1 \leq \delta < \frac{n-1}{n}$. Recall that when $\delta = \delta_1$, constraint 1 in (38) is binding and $v_1 = \alpha_1^{(1)}$. Because v_1 is strictly increasing in δ, if $\delta < \delta_1$ then $v_1 < \alpha_1^{(1)}$, leading to violation of constraint 1. Similarly, if $\delta > \delta_1$ then $v_1 > \alpha_1^{(1)}$ and all constraints are satisfied.

14) Next, we characterise the optimal path in terms of values rather than prices. Using constraint k and (11), we obtain $v_k = \frac{n\delta}{n-1} v_1$. Next, consider the constraints for each period s for $s < k$. Noting that $p_s \leq 1$ and using (11) yields $v_s \leq 1/n + \delta v_{s+1}$. Using constraint s and (11) leads to $v_s \leq \frac{\delta \alpha_1^{(s)} n}{\alpha_1^{(s)} n-1} v_{s+1}$. Employing the complementary slackness conditions of 3) then yields $v_s = \min\{1/n + \delta v_{s+1}, \frac{\delta \alpha_1^{(s)} n}{\alpha_1^{(s)} n-1} v_{s+1}\}$. This implies v_s is decreasing in s.

15) Because $v_k = p_k$ and v_s is decreasing in s, this implies $v_s > p_k$. Hence, there is no incentive to undercut to the bottom of the cycle in any period $s < k$.

16) We now show that there is no incentive to undercut by a full cut-off level: $v_s \geq \alpha_1^{(s+1,s)} p_{s+1}$. Using $v_s \geq \frac{\delta \alpha_1^{(s)} n}{\alpha_1^{(s)} n-1} v_{s+1}$ and marginal undercutting constraint $s+1$ yields $v_s \geq \frac{\delta \alpha_1^{(s)} n}{\alpha_1^{(s)} n-1} \alpha_1^{(s+1)} p_{s+1}$. Comparison with the one-step undercutting constraint $v_s \geq \alpha_1^{(s+1,s)} p_{s+1}$ gives the sufficient condition

$$\delta \geq \frac{\alpha_1^{(s)} n - 1}{\alpha_1^{(s)} n} \frac{\alpha_1^{(s+1,s)}}{\alpha_1^{(s+1)}}. \tag{40}$$

The condition $\delta \geq \delta_b$ is thus sufficient to deter all one-step undercutting deviations.

17) We show, inductively, that if $\delta \geq \delta_b$, then there is no incentive to undercut by $q > 1$ steps. Suppose there is no incentive to undercut q steps in any period $s = 1, \ldots, k-q-1$. In period $s+1$, this means $v_{s+1} \geq \alpha_1^{(s+q+1,s+1)} p_{s+q+1}$. Combining with the condition $v_s \geq \frac{\delta \alpha_1^{(s)} n}{\alpha_1^{(s)} n-1} v_{s+1}$ yields $v_s \geq \alpha_1^{(s+q+1,s+1)} \frac{\delta \alpha_1^{(s)} n}{\alpha_1^{(s)} n-1} p_{s+q+1}$. If this implies that $v_s \geq \alpha_1^{(s+q+1,s)} p_{s+q+1}$, then a $q+1$-step undercut is deterred. A sufficient condition is therefore $\delta \alpha_1^{(s+q+1,s+1)} \frac{\alpha_1^{(s)} n}{\alpha_1^{(s)} n-1} \geq \alpha_1^{(s+q+1,s)}$ or

$$\delta \geq \frac{\alpha_1^{(s+q+1,s)}}{\alpha_1^{(s+q+1,s+1)}} \frac{\alpha_1^{(s)} n - 1}{\alpha_1^{(s)} n}. \tag{41}$$

The one-step condition (40) at period $s+q$ implies (41) if

$$\begin{aligned} \frac{\alpha_1^{(s+1,s)}}{\alpha_1^{(s+1)}} &\geq \frac{\alpha_1^{(s+q+1,s)}}{\alpha_1^{(s+q+1,s+1)}} \\ \Leftrightarrow \frac{\alpha_1^{(s+q+1,s+1)}}{\alpha_1^{(s+q+1,s)}} - 1 &\geq \frac{\alpha_1^{(s+1)}}{\alpha_1^{(s+1,s)}} - 1 \\ \Leftrightarrow \frac{(1-\beta^{(s+q+1)})((1-\beta^{(s+1)})^{n-1} - (1-\beta^{(s)})^{n-1})}{n\beta^{(s+q+1)} + (1-\beta^{(s+q+1)})(1-\beta^{(s)})^{n-1}} &\geq \frac{(1-\beta^{(s+1)})((1-\beta^{(s+1)})^{n-1} - (1-\beta^{(s)})^{n-1})}{n\beta^{(s+1)} + (1-\beta^{(s+1)})(1-\beta^{(s)})^{n-1}} \\ \Leftrightarrow n\beta^{(s+1)}(1-\beta^{(s+q+1)}) &\leq n\beta^{(s+q+1)}(1-\beta^{(s+1)}), \end{aligned}$$

where the final inequality holds because $\beta^{(s+q+1)} \geq \beta^{(s+1)}$. Hence, there is no incentive for a $q+1$-step undercut. Applying the argument inductively, there is no incentive for any i-step undercut for $i \geq 1$. □

Proof of Proposition 4

Proof. Describe a generic two-firm asymmetric path by the pair of price paths $\{p_s^1\}_{s=1}^{\infty}$, $\{p_s^2\}_{s=1}^{\infty}$, where p_s^1 and p_s^2 are the prices charged by Firms 1 and 2 in period s, respectively. Introduce a derivative symmetric path $\{p_s^3\}_{s=1}^{\infty}$, defined as follows. In periods that contain no supporting prices, let $p_s^3 = (p_s^1 + p_s^2)/2$. In periods containing a supporting price, set $p_s^3 = \min\{p_s^1, p_s^2\}$. Define the associated profits and continuation values in period s for Firm j as π_s^j and v_s^j, respectively, where Firm 3 indicates a representative firm on the symmetric path.

We first establish that the value of the symmetric path is greater than the average value of the asymmetric price paths for any period s. In periods containing no supporting prices, the average profit of the asymmetric path is

$$(\pi_s^1 + \pi_s^2)/2 = (\alpha \min\{p_s^1, p_s^2\} + \gamma \max\{p_s^1, p_s^2\})/2,$$

while the profit of the symmetric path is

$$\pi_s^3 = (p_s^1 + p_s^2)/4.$$

Because $\alpha > \gamma$, it follows that the profit of the symmetric path is greater. In periods containing a supporting price, average profits for both symmetric and asymmetric paths are the same:

$$(\pi_s^1 + \pi_s^2)/2 = \pi_s^3 = \min\{p_s^1, p_s^2\}.$$

Combining all periods, it then follows that

$$v_s^3 \geq (v_s^1 + v_s^2)/2. \tag{42}$$

Next, we verify that all incentive constraints are satisfied for the symmetric path if they are satisfied for each asymmetric path. Symmetric paths must satisfy three types of constraint: marginal undercutting, undercutting to the supporting price, and relenting. First, consider periods with no contemporaneous supporting prices and recall that $v_s^3 \geq (v_s^1 + v_s^2)/2$ and $p_s^3 = (p_s^1 + p_s^2)/2$. If the marginal undercutting constraints are met for both asymmetric paths then $v_s^j \geq \alpha p_s^j$, for $j = 1,2$. It follows that $v_s^3 \geq \alpha p_s^3$, and the marginal undercutting constraint is satisfied for the symmetric path. Let the supporting price in period s be p. To prevent undercutting below the supporting price for the asymmetric paths requires $v_s^j \geq p$ for $j = 1,2$, which implies that $v_s^3 \geq p$. Hence, undercutting to the supporting price is deterred in the symmetric path. By assumption, search costs are sufficiently high that the reservation price is no lower than the choke price. Hence, the value of an upward price deviation is equal to γ for all paths. The relenting constraints for the asymmetric paths are then $v_s^j \geq \gamma$ for $j = 1,2$, which implies that $v_s^3 \geq \gamma$. Hence, the relenting constraint is satisfied for the symmetric path.

Second, consider incentive constraints for periods with a supporting price. Recall that $v_s^3 \geq (v_s^1 + v_s^2)/2$ and $p_s^3 = \min\{p_s^1, p_s^2\}$. To prevent undercutting below the supporting price for the asymmetric paths requires $v_s^j \geq \min\{p_s^1, p_s^2\}$, for $j = 1,2$, which implies that $v_s^3 \geq \min\{p_s^1, p_s^2\}$. Hence, there is no incentive to undercut to the supporting price on the symmetric path. Next, notice that there are no separate marginal undercutting constraints on the symmetric path in periods with a supporting price. The relenting constraints for the asymmetric paths are again $v_s^j \geq \gamma$ for $j = 1,2$, which implies that $v_s^3 \geq \gamma$. Hence, the relenting constraint is satisfied for the symmetric path.

The above arguments establish that for every asymmetric path, there exists a symmetric path with weakly higher average value that satisfies all incentive constraints. If there exists a period s with no supporting prices in which $p_s^1 \neq p_s^2$, then the inequality (42) is strict. Suppose instead that the asymmetric path involves $p_s^1 \neq p_s^2$ only in periods with a supporting price. In this case, there exists s with $v_s^1 \neq v_s^2$ on the asymmetric path. To prevent undercutting below the supporting price requires $\min\{v_s^1, v_s^2\} \geq \min\{p_s^1, p_s^2\}$. On the optimal path, this constraint must be binding. For the symmetric path, $v_s^3 > \min\{v_s^1, v_s^2\}$. Hence, there is scope to raise p_s^3 without violating the marginal undercutting constraint on the symmetric path. Therefore, there exists a symmetric path with higher average value that satisfies all incentive constraints. □

Proof of Proposition 5

Proof. Redefine the cartel value

$$v_1 = \frac{(1-\theta)p_1 + \delta(1-\theta)p_2 + \cdots + (1-\theta+k\theta)p_k}{n(1-\delta^k)},$$

and define v_s, $s = 2,\ldots,k$ analogously. Applying the same steps as Lemma 3, the cartel's problem becomes

$$\max_{p_1\ldots p_k\in[0,1]} v \tag{43}$$

subject to

$$\begin{aligned}
&v_s \geq (1-\theta)\alpha_1 p_s,\ s = 1,\ldots,k-1, \text{ and } v_k \geq (1-\theta+k\theta)p_k,\\
&v_s \geq (1-\theta+s\theta)p_k,\ s = 1,\ldots,k-1,\\
&v_s \geq (1-\theta)\alpha_n \min\{1, z_s\},\ s = 1,\ldots,k-1.
\end{aligned}$$

Let

$$\hat{v}_s = \frac{v_s}{1-\theta},\ s = 1,\ldots,k, \qquad \hat{p}_s = p_s,\ s = 1,\ldots,k-1, \qquad \hat{p}_k = \left(1 + \frac{k\theta}{1-\theta}\right)p_k, \tag{44}$$

and rewrite the constraints

$$\hat{v}_s \geq \alpha_1 \hat{p}_s,\ s = 1,\ldots,k-1, \text{ and } \hat{v}_k \geq \hat{p}_k, \tag{45}$$

$$\hat{v}_s \geq \left(1 - \frac{(k-s)\theta}{1-\theta+k\theta}\right)\hat{p}_k,\ s = 1,\ldots,k-1, \tag{46}$$

$$\hat{v}_s \geq \alpha_n \min\{1, z_s\},\ s = 1,\ldots,k-1, \tag{47}$$

Comparison of equations (13) - (15) and (45) - (47) reveals that only the constraints in (46) have changed. These constraints were not binding in the original problem. They are now easier to satisfy, so they will not be binding in the current problem. Hence,

$$p_s = p_s^0,\ s = 1,\ldots,k-1, \qquad p_k = \left(1 + \frac{k\theta}{1-\theta}\right)^{-1} p_k^0,$$

as required. Further, $v_s = (1-\theta)v_s^0$, $s = 1,\ldots,k$.

□

B Profits and punishment

An important implication of our model is that if consumers are imperfectly attentive, it is possible that price dispersion could make collusion easier to sustain. The mechanism is quite intuitive. Price variation makes it harder for consumers to detect modest price undercutting. This affects the sustainability of collusion by reducing the payoffs to firms deviating from cartel policies. However, if price dispersion is effective at obfuscating the price process, it is also possible that the ability of the cartel to punish a defector is impacted. In this section, we examine the trade-off involved if this is the case.

First, we introduce some additional notation and definitions.[25] Let $\mathbf{p} \equiv (p_1, p_2, \ldots, p_n)$ denote a price vector, and $\mathbf{p}_j$ the associated vector with p_j removed. Let $\vec{\mathbf{p}} \equiv (p, p, \ldots, p)$ describe a vector of n prices in which each firm chooses the same price p. Let $\vec{\mathbf{p}}_j$ be the corresponding $n-1$ price vector with the price of firm j omitted. We generalise the model in the body of the paper by allowing attention to be triggered by unusually low *and* high prices. Thus,

$$\phi\left(p, \underline{p}, \overline{p}\right) = \begin{cases} \beta & \text{if } p \in (\underline{p}, \overline{p}); \\ 1 & \text{otherwise,} \end{cases}$$

where $\underline{p}$ and $\overline{p}$ are the lower and upper bounds, respectively. Profits for firm j are then given by $\pi_j(\mathbf{p}; \underline{p}, \overline{p}) = R(p_j) s_j(\mathbf{p}; \underline{p}, \overline{p})$. Define $\pi_j^*(\mathbf{p}_j; \underline{p}, \overline{p}) \equiv \sup_{p'} \pi_j(p', \mathbf{p}_j; \underline{p}, \overline{p})$. Letting $\check{\mathbf{p}}_j$ minimise $\pi_j^*(\mathbf{p}_j; \underline{p}, \overline{p})$, minmax profits for j are then given by $\pi_j^*(\check{\mathbf{p}}_j; \underline{p}, \overline{p})$.

We adopt the following restrictions on the firm's profit function.

Assumption 1. *For all j and $k \neq j$,*

$$\frac{\partial s_j(\mathbf{p}; \underline{p}, \overline{p})}{\partial p_k} \geq 0.$$

Assumption 2. *For all j and all p,*

$$s_j(\vec{\mathbf{p}}, \underline{p}, \overline{p}) = \frac{1}{n}.$$

Assumption 3. *There exists $\tilde{p} \leq \underline{p}$ such that for all $p \in [\tilde{p}, \overline{p}]$ and all j,*

$$\pi_j(\vec{\mathbf{p}}; \underline{p}, \overline{p}) > \pi_j^*(\vec{\mathbf{0}}_j; \underline{p}, \overline{p}).$$

Assumption 4. *For all j and all $p < \hat{p}$,*

$$\pi_j^*(\vec{\mathbf{p}}_j; \underline{p}, \overline{p}) - \pi_j(\vec{\mathbf{p}}; \underline{p}, \overline{p})$$

is nondecreasing in p.

[25] Our exposition is closely related to Lambson (1987).

Assumption 1 imposes monotonicity and implies that marginal cost pricing by all other firms minmaxes firm j's profits. By Assumption 2, the market is shared equally if all firms set the same price. Assumption 3 ensures that it is possible for firms to earn collusive profits that are greater than the minmax profit level. Assumption 4 suggests that the returns from deviation are weakly increasing in prices.

A *punishment* for firm j is a sequence of price vectors $\tau_j = \{\mathbf{p}(t,j)\}_{t=0}^{\infty}$, and a *simple penal code* is a vector of punishments $\tau = (\tau_1, \tau_2, \ldots, \tau_n)$. The value to firm j if the punishment for firm i is followed is $V_j(\tau_i; \underline{p}, \overline{p}) = \sum_{t=0}^{\infty} \delta^t \pi_j(\mathbf{p}(t,i); \underline{p}, \overline{p})$. A simple penal code is *credible* if $\forall i, j$, and t',

$$\sum_{t=0}^{\infty} \delta^t \pi_j(\mathbf{p}(t'+t, i); \underline{p}, \overline{p}) \geq \pi_j^*(\mathbf{p}_j(t', i); \underline{p}, \overline{p}) + \delta V_j(\tau_j; \underline{p}, \overline{p}). \tag{48}$$

If Firm j receives minmax profits in every period, she obtains her *security value* $\underline{v}(\underline{p}, \overline{p}) \equiv \pi_j^*(\breve{\mathbf{p}}_j; \underline{p}, \overline{p})/(1-\delta)$. For any credible penal code τ, and for all j, we must have $V_j(\tau_j; \underline{p}, \overline{p}) \geq \underline{v}(\underline{p}, \overline{p})$. Define a security level punishment $\underline{\tau}_j$ as the punishment that achieves the value $V_j(\underline{\tau}_j; \underline{p}, \overline{p}) = \underline{v}(\underline{p}, \overline{p})$, and the associated simple penal code as $\underline{\tau}$. An *optimal penal code* is a simple penal code that minimises $V_j(\tau_j, \underline{p}, \overline{p})$, subject to the constraint that it is credible.

Consider the infinitely repeated two period cycle $\{p_a, p_b\}$ where $p_a, p_b \in [\underline{p}, \overline{p}]$ and $\underline{p} \geq \tilde{p}$.[26] A fixed price is a special case in which $p_a = p_b$. Let $\Phi(\underline{p}, \overline{p})$ be the set of two-period cycles *sustainable* by $\underline{\tau}$. Then, $\{p_a, p_b\} \in \Phi(\underline{p}, \overline{p})$ if and only if

$$\frac{\pi_j(\vec{\mathbf{p}}_a; \underline{p}, \overline{p}) + \delta \pi_j(\vec{\mathbf{p}}_b; \underline{p}, \overline{p})}{1+\delta} \geq (1-\delta)\pi_j^*(\vec{\mathbf{p}}_{a,j}; \underline{p}, \overline{p}) + \delta \pi_j^*(\vec{\mathbf{0}}_j; \underline{p}, \overline{p}), \tag{49}$$

$$\frac{\pi_j(\vec{\mathbf{p}}_b; \underline{p}, \overline{p}) + \delta \pi_j(\vec{\mathbf{p}}_a; \underline{p}, \overline{p})}{1+\delta} \geq (1-\delta)\pi_j^*(\vec{\mathbf{p}}_{b,j}; \underline{p}, \overline{p}) + \delta \pi_j^*(\vec{\mathbf{0}}_j; \underline{p}, \overline{p}). \tag{50}$$

Proposition 6. *Suppose Assumptions 1-4 hold. Fix $p_a > \tilde{p}$ such that $\{p_a, p_a\} \in \Phi(p_a, p_a)$ and let $\delta^*(p_a)$ be the solution to*

$$\pi_j(\vec{\mathbf{p}}_a; p_a, p_a) = (1-\delta^*(p_a))\pi_j^*(\vec{\mathbf{p}}_{a,j}; p_a, p_a) + \delta^*(p_a)\pi_j^*(\vec{\mathbf{0}}_j; p_a, p_a). \tag{51}$$

Then there exists a price $p_b < p_a$ and a discount factor $\delta < \delta^*(p_a)$ such that $\{p_a, p_b\} \in \Phi(p_b, p_a)$ if equations (52) and (53) hold:

[26]The extension to a finite k period cycle is immediate.

$$\frac{\delta^*(p_a)}{1+\delta^*(p_a)}\left(\pi_j(\vec{\mathbf{p}}_b; p_b, p_a) - \pi_j(\vec{\mathbf{p}}_a; p_a, p_a)\right) > \tag{52}$$
$$(1-\delta^*(p_a))\left(\pi_j^*(\vec{\mathbf{p}}_{a,j}; p_b, p_a) - \pi_j^*(\vec{\mathbf{p}}_{a,j}; p_a, p_a)\right) + \delta^*(p_a)\left(\pi_j^*(\vec{\mathbf{0}}_j; p_b, p_a) - \pi_j^*(\vec{\mathbf{0}}_j; p_a, p_a)\right),$$
$$\frac{1}{1+\delta^*(p_a)}\left(\pi_j(\vec{\mathbf{p}}_b; p_b, p_a) - \pi_j(\vec{\mathbf{p}}_a; p_a, p_a)\right) > \tag{53}$$
$$(1-\delta^*(p_a))\left(\pi_j^*(\vec{\mathbf{p}}_{b,j}; p_b, p_a) - \pi_j^*(\vec{\mathbf{p}}_{a,j}; p_a, p_a)\right) + \delta^*(p_a)\left(\pi_j^*(\vec{\mathbf{0}}_j; p_b, p_a) - \pi_j^*(\vec{\mathbf{0}}_j; p_a, p_a)\right).$$

Proof. 1. By Assumption 1, security level payoffs are given by

$$\underline{v}(\underline{p}, \overline{p}) = \frac{\pi_j^*(\vec{\mathbf{0}}_j; \underline{p}, \overline{p})}{1-\delta}. \tag{54}$$

2. Let $\delta^*(p_a, p_b; \underline{p}, \overline{p})$ be the lowest discount factor satisfying (49) and (50). Notice that the left hand side of (50) is increasing in δ and the right hand side of (50) is decreasing in δ. Therefore, (50) is satisfied for $\delta \geq \delta^*(p_a, p_b; \underline{p}, \overline{p})$.

Differentiating (49) by δ, the following condition ensures that (49) will also be satisfied for $\delta \geq \delta^*(p_a, p_b; \underline{p}, \overline{p})$:

$$\frac{\pi_j(\vec{\mathbf{p}}_b; \underline{p}, \overline{p}) - \pi_j(\vec{\mathbf{p}}_a; \underline{p}, \overline{p})}{(1+\delta)^2} > \pi_j^*(\vec{\mathbf{0}}_j; \underline{p}, \overline{p}) - \pi_j^*(\vec{\mathbf{p}}_{a,j}; \underline{p}, \overline{p}). \tag{55}$$

We can always choose p_b sufficiently close to p_a to satisfy (55). Combining these results, there exists $p_b < p_a$ such that (49) and (50) are satisfied for $\delta \geq \delta^*(p_a, p_b; \underline{p}, \overline{p})$.
3. Next, we construct an optimal penal code $\underline{\tau}$. To do so, first define the price p^u and integer u such that

$$\sum_{t=0}^{u-2} \delta^t \pi_j(\vec{\mathbf{0}}; \underline{p}, \overline{p}) + \delta^{u-1}\pi_j(\vec{\mathbf{p}}^u; \underline{p}, \overline{p}) + \delta^u \frac{\pi_j(\vec{\mathbf{p}}_a; \underline{p}, \overline{p}) + \delta\pi_j(\vec{\mathbf{p}}_b; \underline{p}, \overline{p})}{1-\delta^2} = \underline{v}(\underline{p}, \overline{p}). \tag{56}$$

There are two possibilities for the relationship between minmax profits and profits in the initial, harshest, phase of punishment. If $\pi_j(\vec{\mathbf{0}}; \underline{p}, \overline{p}) = \pi_j^*(\vec{\mathbf{0}}_j; \underline{p}, \overline{p})$, then there is a Nash equilibrium to the stage game that minmaxes each player. In this case, we use infinite repetition of this stage game Nash equilibrium as punishment by setting $u = \infty$. Note that this was the situation we examined in the body of the paper.

Alternatively, if $\pi_j(\vec{\mathbf{0}}; \underline{p}, \overline{p}) < \pi_j^*(\vec{\mathbf{0}}_j; \underline{p}, \overline{p})$, then we must show that there exists an integer u and a price $p^u < p_b$ that satisfies (56). Initially suppose $p_b = p_a$. Next, note that Assumption 3 implies that $\pi_j(\vec{\mathbf{p}}_b; \underline{p}, \overline{p}) > \pi_j^*(\vec{\mathbf{0}}_j; \underline{p}, \overline{p})$. It follows that there must exist a pair u and $p^u < p_a$ that satisfies (56) in this case. Therefore, there exists a $p_b < p_a$ such that this condition is also satisfied.

For $\{p_a, p_b\} \in \Phi(\underline{p}, \overline{p})$ and each j, then define $\underline{\tau}_j$ as follows

$$\mathbf{p}(t,i) = \begin{cases} \vec{\mathbf{0}} & 0 \le t \le u-2 \\ \vec{\mathbf{p}}^u & t = u-1 \\ \{\vec{\mathbf{p}}_a, \vec{\mathbf{p}}_b\} & t \ge u. \end{cases}$$

Note that we must have $u \ge 1$. Otherwise, the penal code consists only of the infinite price cycle $\{p_a, p_b\}$ which has a value greater than the security value by Assumption 3. In points 4. and 5., we consider the case where $u \ge 2$. In 6., we consider $u = 1$.

4. Let us show credibility of the optimal penal code $\underline{\tau}$ for the case $u \ge 2$ by establishing (48). In period $t' \le u-2$, we can rewrite (48) as

$$\sum_{t=0}^{u-2-t'} \delta^t \pi_j(\vec{\mathbf{0}}; \underline{p}, \overline{p}) + \delta^{u-1-t'} \pi_j(\vec{\mathbf{p}}^u; \underline{p}, \overline{p}) + \delta^{u-t'} \frac{\pi_j(\vec{\mathbf{p}}_a; \underline{p}, \overline{p}) + \delta \pi_j(\vec{\mathbf{p}}_b; \underline{p}, \overline{p})}{1-\delta^2} \ge \pi_j^*(\vec{\mathbf{0}}_j; \underline{p}, \overline{p}) + \delta \underline{v}(\underline{p}, \overline{p}) \tag{57}$$

For $t' = 0$, by (56), equation (57) is satisfied with equality. For $0 < t' \le u-2$, because $\pi_j(\vec{\mathbf{0}}; \underline{p}, \overline{p}) < \min\{\pi_j(\vec{\mathbf{p}}^u; \underline{p}, \overline{p}), \pi_j(\vec{\mathbf{p}}_a; \underline{p}, \overline{p}), \pi_j(\vec{\mathbf{p}}_b; \underline{p}, \overline{p})\}$, (57) must also be satisfied.

For $t' \ge u$, inequality (48) follows if $\{p_a, p_b\} \in \Phi(p_b, p_a)$. We establish the conditions for this result in step 5, below.

Finally, for $t' = u-1$, condition (48) requires

$$\pi_j(\vec{\mathbf{p}}^u; \underline{p}, \overline{p}) + \delta \frac{\pi_j(\vec{\mathbf{p}}_a; \underline{p}, \overline{p}) + \delta \pi_j(\vec{\mathbf{p}}_b; \underline{p}, \overline{p})}{1-\delta^2} \ge \pi_j^*(\vec{\mathbf{p}}_j^u; \underline{p}, \overline{p}) + \delta \underline{v}(\underline{p}, \overline{p}) \tag{58}$$

Notice that if $\{p_a, p_b\} \in \Phi(p_b, p_a)$, then from (50)

$$\pi_j(\vec{\mathbf{p}}_b; \underline{p}, \overline{p}) + \delta \frac{\pi_j(\vec{\mathbf{p}}_a; \underline{p}, \overline{p}) + \delta \pi_j(\vec{\mathbf{p}}_b; \underline{p}, \overline{p})}{1-\delta^2} \ge \pi_j^*(\vec{\mathbf{p}}_{b,j}; \underline{p}, \overline{p}) + \delta \underline{v}(\underline{p}, \overline{p}) \tag{59}$$

Because $p^u > p_b$, Assumption 4 and (59) imply (58).

5. Now, for $u \ge 2$, we compare the sustainability of a fixed price path with that of a cycle.

Consider first a candidate fixed price equilibrium with price p_a. In the candidate equilibrium, $\underline{p} = \overline{p} = p_a$. Consequently p_a is sustainable if and only if

$$\frac{\pi_j(\vec{\mathbf{p}}_a; p_a, p_a)}{1-\delta} \ge \pi_j^*(\vec{\mathbf{p}}_{a,j}; p_a, p_a) + \frac{\delta}{1-\delta} \pi_j^*(\vec{\mathbf{0}}_j; p_a, p_a). \tag{60}$$

Let $\delta^*(p_a)$ solve (60) with equality and rearrange to obtain (51).

Next consider a candidate price cycle equilibrium with prices p_a and p_b for some $p_b < p_a$. In the candidate equilibrium, $\underline{p} = p_b$ and $\overline{p} = p_a$. The cycle $\{p_a, p_b\}$ is sustainable if and only if

$$\frac{\pi_j(\vec{\mathbf{p}}_a; p_b, p_a) + \delta\pi_j(\vec{\mathbf{p}}_b; p_b, p_a)}{1+\delta} \geq (1-\delta)\pi_j^*(\vec{\mathbf{p}}_{a,j}; p_b, p_a) + \delta\pi_j^*(\vec{\mathbf{0}}_j; p_b, p_a), \tag{61}$$

$$\frac{\pi_j(\vec{\mathbf{p}}_b; p_b, p_a) + \delta\pi_j(\vec{\mathbf{p}}_a; p_b, p_a)}{1+\delta} \geq (1-\delta)\pi_j^*(\vec{\mathbf{p}}_{b,j}; p_b, p_a) + \delta\pi_j^*(\vec{\mathbf{0}}_j; p_b, p_a). \tag{62}$$

The candidate price cycle equilibrium is then sustainable for a greater range of discount factors if the inequalities in (61) and (62) are strict for $\delta = \delta^*(p_a)$. Combining (51), (61) and (62), and applying Assumption 2 we obtain the conditions (52) and (53).

6. Finally, consider the case $u = 1$. Credibility at time $t = 0$ is established by the same argument examined for $t' = u - 1$ in point 4. Credibility at time $t \geq 1$ and sustainability are both demonstrated by repeating the argument of point 5. □

Proposition 6 establishes the conditions under which collusion is supportable for a greater range of discount factors using a price cycle rather than a fixed price. Equations (52) and (53) highlight the trade-off between the profitability of deviation and punishment across a cycle and a fixed price. The left hand side of each equation measures the difference between the average value of collusion with a cycle and with a fixed price. Because the cycle involves spending some time setting the lower price p_b, average profitability under the cycle path will be lower. However, this difference will be (i) small if p_b is not too far below p_a; and (ii) less important if we consider cycles of greater length.

The first term on the right hand side is the difference between deviation profits under a cycle and a fixed price. Our argument in the body of the paper is that the profits from deviation may be markedly higher in the context of a fixed price path relative to a price cycle because price dispersion makes it difficult for consumers to distinguish small price differences that lie within their realm of experience. By this argument, we may expect this term to be negative and substantial.

The second term on the right hand side is the difference between punishment profits under the cycle and the fixed price path. In our specification of Section 2, a zero profit stage game equilibrium was available and this term dropped out. Here, we allow punishment payoffs to depend on the extent of price dispersion. Intuitively though, the influence of price dispersion is likely to be small. To illustrate, suppose firms have enjoyed a spell of collusion. One firm then deviates, triggering a price war in which all firms set price equal to marginal cost for a sustained period. This price war represents a sharp break in the pricing pattern of the market and is likely to attract the attention of consumers regardless of the precise form of collusive pricing. If a single firm were to depart from the prescribed punishment by raising price, it is likely that consumers will notice that this firm offers a price that is above the price of its rivals. This argument pertains whether collusion had taken the form of a fixed price or a price cycle.

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