Economics Working Paper Series

2018 - 9

Testing for Stationarity at High Frequency

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July 2018

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July 16, 2018

Abstract

The high frequency behavior of the KPSS test, which is most commonly used to test for stationarity, is analyzed in a continuous time framework. Our asymptotics show that the test has no discriminatory power at high frequency: It either always rejects stationarity or has no nontrivial power at high frequency. The test becomes valid at high frequency only when the bandwidth of its longrun variance estimate is chosen suitably in our framework. We also analyze the residual-based KPSS test for cointegration.

JEL Classification: C13, C22

Keywords: KPSS test, testing for stationarity, testing for cointegration, continuous time process, high frequency observation

*We would like to thank Yoosoon Chang, Shin Kanaya and Jihyun Kim for helpful discussions, and to the participants at SETA2015 (Tokyo) and EC2 2017 (Amsterdam), and the seminar attendants at Indiana University and Pennsylvania State University.

1. Introduction

The test proposed in Kwiatkowski et al. (1992), often referred to as the KPSS test, has been used most extensively to test for stationarity of a time series. It relies on cumulation of squared partial sums of the demeaned and/or detrended series with a correction for autocorrelation using a nonparametric longrun variance estimate. There have been many variants of the KPSS test introduced in the literature. For example, the MR/S test of Lo (1991) is similar to the KPSS test except that it is based on the range of the partial sum of residuals. Another variant of the KPSS test is the indicator KPSS test of de Jong et al. (2007), who calculate the same statistic as the KPSS test but relying on the data that is given as an indicator of whether the original data point is above or below the median. The indicator KPSS test is shown to be robust to fat tailed distributions. Other variants of the KPSS test include Giraitis et al. (2003). In practice, the KPSS test is also very useful when it is applied to the test for cointegration in a set of nonstationary time series. See, e.g., Shin (1994). The asymptotic theories of the KPSS test and its variants are well developed.

Due to the abundance of observations available at high frequency, it has become increasingly more popular to use high frequency data in various econometric practice. Testing for stationarity is not an exception and, recently, the KPSS test and its variants are often applied by many to observations at daily or higher frequency. However, at high frequency, the tests behave rather differently from what their existing asymptotic theories predict. It does not appear that their existing asymptotics are useful at all. In fact, we analyze the forward premium of US/UK exchange rates and US treasury bill rates and illustrate that the tests are highly sensitive to the sampling frequency, as well as the bandwidth choice required for the estimation of longrun variance used in the tests. In fact, for some common choices of bandwidth, the values of test statistics increase very rapidly as the sampling frequency exceeds the monthly level and almost explodes at the daily level. It is therefore necessary to develop a new set of asymptotics that are useful to explain and predict the behaviors of the KPSS test and its variants at high frequency.

In the paper, we introduce a novel continuous time framework and develop a new set of asymptotics for the KPSS type tests based on high frequency data. Our framework and asymptotics are applicable for both the stationarity test and the residual based test for cointegration.[1] In our asymptotics, we assume that samples are drawn discretely from

[1]Busetti and Taylor (2005) studied the locally best invariant (LBI) stationarity tests, and discussed the sampling frequency effect using a continuous time model under the infill fixed span asymptotics. They show that the high-frequency power property of the test is different for stock and flow variables. In this paper, we develop a more general continuous time framework, and analyze the KPSS type of test which is the longrun variance corrected LBI test; while the distinction between stock and flow variables is not the focus here. In terms of asymptotics, we consider both infill and long span asymptotics. In fact, the long span analysis is

underlying continuous time processes, and that the sampling interval δ shrinks down to zero as well as the sample span T increases up to infinity. To obtain the asymptotics that are more relevant to the tests applied for observations at high frequency, we let $\delta \to 0$ fast enough relative to $T \to \infty$. The resulting asymptotics are found to be much useful, explaining and predicting the behaviors of the KPSS type tests at high frequency very well. In fact, our asymptotics show that the tests do not have proper discriminatory powers at high frequency if they are implemented with the usual longrun variance estimators obtained in the usual discrete time framework. The tests either always reject stationarity or have no nontrivial power at high frequency, depending upon bandwidth choices.

Our asymptotics also show that we only need to choose the bandwidth in continuous time framework to make the KPSS type tests valid and work as expected at high frequency. The required modifications are therefore truly simple. Once modified by choosing the bandwidth properly, the tests have correct sizes and perfect powers asymptotically. Our simulation results also show that the tests developed in continuous time framework are no longer sensitive to the sampling frequency and perform as expected in finite samples. In fact, the tests using high frequency observations are generally much more stable than the ones using low frequency observations. This implies that high frequency observations should always be used whenever they are available. The continuous time KPSS type tests for stationarity have the limit null distributions that are exactly the same as those for the conventional tests developed in discrete time framework. However, our continuous time KPSS type tests for cointegration have limit null distributions, which are generally model dependent and distinct from the limit null distributions of the discrete time KPSS type tests. This is simply because we allow for a much more general class of nonstationary processes in our continuous time framework. To deal with the model dependence of the critical values of the tests, we propose to use a subsampling method. The subsample test appears to work reasonably well according to our simulations.

Finally, our continuous time KPSS type tests developed in the paper are comparable to other existing tests for stationarity and nonstationarity of a continuous time process. Our tests are widely applicable for testing the stationarity of a continuous time process or for the presence of cointegration in a set of nonstationary continuous time processes, since no specific assumptions on the structures of underlying continuous time processes are made in our asymptotics. In fact, not much research has yet been done on testing for stationarity and nonstationarity in continuous time framework. Only recently, Bandi and Corradi (2014) and Kanaya (2011) proposes a nonparametric testing approach. The method proposed in Bandi and Corradi (2014) uses nonstationarity as the null hypothesis and it is only applicable for

essential and more sensible to analyze the properties of stationarity tests.

nonstationary diffusions, an important but special class of continuous time nonstationary processes. The KPSS test is more directly comparable to that of Kanaya (2011) as the test of stationarity for a continuous time process. The former, however, appears to have substantially better discriminatory power than the latter.

The rest of the paper is organized as follows. Section 2 introduces the KPSS test and the bandwidth selection rules that are commonly used for longrun variance estimation. Some motivational illustrations are also included. Section 3 presents the asymptotics of the KPSS test for stationarity, which are developed in continuous time framework. We show that the KPSS test fails to work if the bandwidth is selected in discrete time framework, and then explain what we need to do to make it valid and work properly. Subsequently, in Section 4, we develop the asymptotics of the residual based KPSS test for cointegration. A subsample test is also introduced and analyzed. In Section 5, we present the asymptotics of several variants of KPSS test. Section 6 concludes the paper. All detailed mathematical proofs are provided in Appendix.

A word on notation. The standard notations such as $\to_p$ and $\to_d$ are used extensively to refer the convergences in probability and in distribution, respectively. We denote by $[U]_T$ the quadratic variation of a continuous time process U over time interval $[0, T]$. Also, $P \sim_p Q$ and $P \prec_p Q$ respectively signify $P = Q(1 + o_p(1))$ and $P/Q = o_p(1)$. Moreover, $P \approx Q$ just implies that we approximate P by Q, and it does not have any precise mathematical meaning in regard to the proximity between P and Q.

2. Background and Preliminaries

For a given time series (u_i), $i = 1, \ldots, n$, the KPSS test statistic is defined as

$$\lambda_n = \frac{1}{n^2 \omega_n^2} \sum_{i=1}^{n} \left(\sum_{j=1}^{i} u_j \right)^2, \tag{2.1}$$

where ω_n^2 is the longrun variance estimate of (u_i). We write

$$\omega_n^2 = \sum_{|j| \le n} K\left(\frac{j}{b_n}\right) \gamma_n(j), \tag{2.2}$$

where $K : \mathbb{R} \to [-1, 1]$ is a kernel function, $\gamma_n(j) = n^{-1} \sum_i u_i u_{i-j}$ is the sample autocovariance function,[2] and b_n is the bandwidth parameter which scales the shape of the kernel

[2]Here and elsewhere in the paper, we assume that the summation in the definition of γ_n runs only over the range where both indices i and $i - j$ are between 1 and n.

function.

The KPSS test is often applied to the fitted residuals from regressions $y_i = \alpha + u_i$ and $y_i = \alpha + \beta x_i + u_i$, where (y_i) and (x_i) are some observed time series. The residuals are given as

$$u_{ni} = u_i - \bar{u}_n \tag{2.3}$$

$$u_{ni} = (u_i - \bar{u}_n) - \sum_{j=1}^{n} u_j(x_j - \bar{x}_n) \left(\sum_{j=1}^{n} (x_j - \bar{x}_n)^2 \right)^{-1} (x_i - \bar{x}_n) \tag{2.4}$$

respectively, with $\bar{u}_n = n^{-1} \sum_{j=1}^{n} u_j$ and $\bar{x}_n = n^{-1} \sum_{j=1}^{n} x_j$. If λ_n is defined with (u_{ni}) in (2.3) instead of (u_i), the test allows for the presence of nonzero mean in (u_i). If λ_n is defined with (u_{ni}) in (2.4) instead of (u_i), the test can be used to test for the presence of cointegration between (y_i) and (x_i). This was shown earlier by Shin (1994). The test based on the fitted residuals (u_{ni}) in (2.3) or (2.4) will be referred to more specifically as the residual based KPSS test, or the RB-KPSS test, whenever it is necessary to distinguish it from the test based directly on (u_i). The RB-KPSS tests based on (u_{ni}) in (2.3) and (2.4) will be analyzed in a single framework.[3]

For the kernel function K introduced in (2.2), we let

$$\pi(s) = \lim_{x \to 0} \frac{1 - K(x)}{|x|^s}$$

for any nonnegative integer s, and define its characteristic exponent as $r = \max\{s : \pi(s) < \infty\}$. If $\pi(s) = \infty$ for any nonnegative integer s, we set $r = \infty$. The characteristic exponent r is a smoothness measure of the kernel function around zero.

Assumption KF We assume that (i) K is continuous at 0 and all but a finite number of other points, and it is symmetric with $K(0) = 1$ and $\int_{-\infty}^{\infty} K^2(x)dx < \infty$, (ii) K' exists at all but a finite number of points and K' is integrable, and (iii) K has a finite characteristic exponent r.

The conditions (i), (ii) and (iii) in Assumption KF are standard and not stringent, and they are satisfied by virtually all kernels used in practical applications including truncated, Bartlett, Parzen, Tukey-Hanning, and quadratic spectral kernels. For commonly used kernel functions, we have $r = 1$ or 2.

[3] Though we do not explicitly consider in the paper, our subsequent theory is also applicable for the test in the presence of linear time trend with some obvious modifications.

It is well recognized that the choice of bandwidth b_n plays a central role in determining the asymptotic behavior of the KPSS test. For consistency of longrun variance estimate, b_n should satisfy $b_n \to \infty$ and $b_n/n \to \infty$ as $n \to \infty$. The optimal bandwidth b_n, which minimizes the asymptotic mean squared error of the longrun variance estimator, for a general stationary time series (u_i) is given by $b_n = c^*(r)n^{1/(2r+1)}$, where

$$c^*(r) = \left(\frac{r\pi^2(r)}{\iota(K^2)}\theta^2(r)\right)^{1/(2r+1)} \tag{2.5}$$

with

$$\theta(r) = \frac{\sum_j |j|^r \gamma(j)}{\sum_j \gamma(j)}.$$

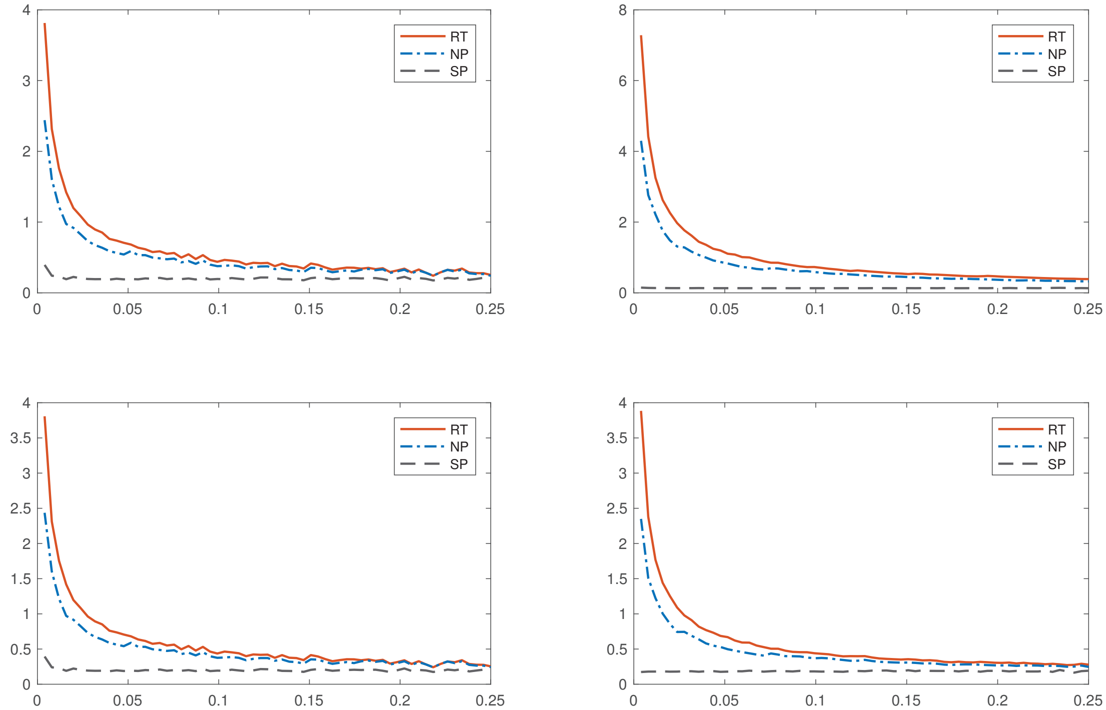
See, e.g., Andrews (1991) for details.[4] Note that the constants $r, \pi(r)$ and $\iota(K^2)$ are all fully determined by the choice of kernel function K. However, the constant $\theta(r)$ depends upon the unknown autocovariance function γ of the underlying time series (u_i).

There are three widely used selection schemes for the bandwidth parameter b_n, which can be summarized as follows. The first scheme, called the rule of thumb (RT), sets $b_n = cn^p$ for some arbitrary constants $c > 0$ and $0 < p < 1$. The other two schemes are data dependent versions of the optimal bandwidth that we may write as $b_n = c_n^*(r)n^{1/(2r+1)}$, where $c_n^*(r)$ is defined analogously as $c^*(r)$ in (2.5) with $\theta(r)$ replaced by its estimate $\theta_n(r)$. We call the second scheme, considered in Newey and West (1994), the nonparametric (NP) scheme, because it uses a nonparametric estimate $\theta_n(r) = \sum_{|j|\leq a_n} |j|^r \gamma_n(j) / \sum_{|j|\leq a_n} \gamma_n(j)$ of $\theta(r)$ with $a_n = cn^p$ for some arbitrary constants $c > 0$ and $0 < p < 1$. The third scheme was proposed and analyzed by Andrews (1991), and will be referred to as the semiparametric (SP) scheme. It relies on the specification of (u_i) as an AR(1) with autoregressive coefficient ρ, in which case we can deduce $\theta(r) = \big[2(1-\rho)/(1+\rho)\big]\rho(r)$, where $\rho(r) = \sum_{j=1}^\infty j^r \rho^j$. Therefore, by simply plugging in the estimated AR coefficient ρ_n in the AR(1) regression of (u_i), we obtain an estimate $\theta_n(r)$ of $\theta(r)$. In particular, when $r = 1$ and 2, we have $\theta_n(1) = 2\rho_n/(1-\rho_n)(1+\rho_n)$ and $\theta_n(2) = 2\rho_n/(1-\rho_n)^2$.

One of the main objectives of our paper is to demonstrate and analyze the dependency of the KPSS test on sampling frequency. Figure 1 shows the values of the KPSS test obtained from the samples collected at various frequencies ranging from quarterly to daily. For our illustration, the 1-month forward and spot US/UK exchange rates are downloaded from the Bank of England over the sample period from January 2, 1979 to June 30, 2015, and the

[4] Needless to say, using an optimal bandwidth minimizing the asymptotic mean squared error of the longrun variance estimator ω_n^2 does not necessarily yield an optimal test based on λ_n in any sense. Optimal bandwidth choices are considered here simply because they are widely used to implement the KPSS test in practice.

Fig. 1. Frequency Dependence of KPSS Test

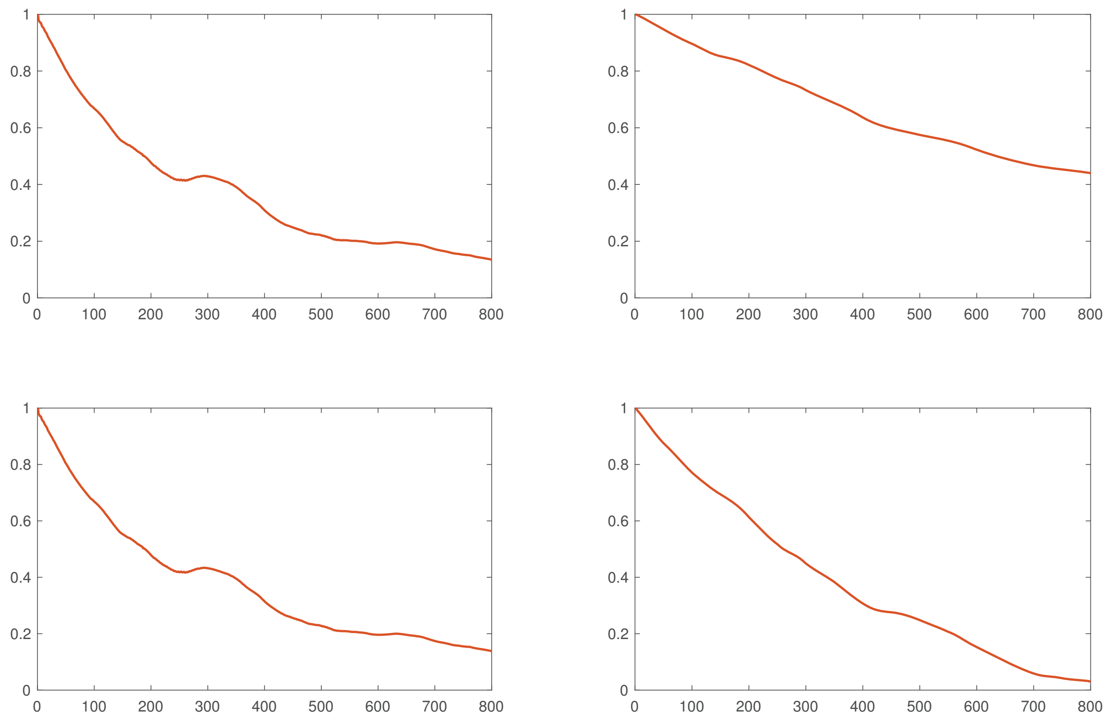


Notes: Presented are the KPSS test values from the samples collected at various frequencies. The vertical axis displays the test values, and the horizontal axis displays the lengths of sampling intervals. We use year as the unit of time, and the sampling intervals given by 1/4 and 1/252 correspond to the quarterly and daily observations, respectively. The top two panels present the stationarity tests for the 1-month forward premium of US/UK exchange rates (left), and for the 3-month US treasury bill rates (right). The lower two panels present the cointegration tests between the 1-month forward US/UK exchange rates and the spot rates (left), and between the 10-year US treasury bond rates and 3-month US treasury bill rates (right). Three bandwidth schemes RT, NP and SP are used with Parzen kernel.

3-month US T-bill rates and 10-year treasury constant maturity rates are downloaded from FRED, St Louis Fed, over the sample period from January 2, 1962 to June 30, 2015, all at daily frequency. The forward premium is defined as the log difference of the 1-month forward and spot exchange rates. We apply the KPSS test to test for stationarity of the forward premium of US/UK exchange rates and the 3-month US T-bill rates, and also to test for the presence of cointegration between the spot and 1-month forward US/UK exchange rates, and between the 3-month US treasury bill and 10-year US treasury bond rates. The results are presented in the upper and lower panels, respectively.

The results for the KPSS test with bandwidth schemes RT and NP are highly sensitive to the sampling frequency, and show some clear pattern as the sampling frequency increases. In fact, the test statistic steadily increases as the sampling frequency increases, and the

Fig. 2. Sample Autocorrelation Function



Notes: Presented are the sample autocorrelation functions (ACF) of the time series we used to compute the KPSS tests for stationarity and cointegration. The top two panels present the sample ACFs of the 1-month forward premium of US/UK exchange rates (left) and the demeaned 3-month US T-bill rates (right). The lower two panels present the sample ACFs of the residuals from the cointegration regression of the 1-month forward US/UK exchange rates on the spot rates (left) and the the 10-year US treasury bond rates on the 3-month US treasury bill rates (right).

increasing trend becomes particularly conspicuous as the sampling frequency exceeds the monthly level. This is true in all cases. Therefore, we are led to reject the null hypothesis of stationarity or the null hypothesis of cointegration decisively in all cases if high frequency observations are used, although the test results tend to be more in favor of the null hypothesis of stationarity or the null hypothesis of cointegration at lower frequencies.[5] On the other hand, the KPSS test with bandwidth scheme SP is robust and it does not show any frequency dependence. As discussed, however, the test is inconsistent, and the frequency robustness comes at a severe cost of its power.

[5] We interpret our result of the KPSS test on stationarity of US/UK forward exchange premium or cointegration between forward and spot US/UK exchange rates, in particular, as an evidence of test failure instead of a new finding. Indeed, as for all other currencies, the time series of US/UK forward and spot exchange rates move so closely each other, and it is quite clear even by an eyeball test that they share a common trend.

To further analyze the frequency dependence of the KPSS test, we plot the sample autocorrelation function (ACF) of the time series we used to compute the KPSS test and test for stationarity or cointegration in Figure 2. For all four time series, the sample ACF increases and approaches unity as the sampling frequency increases. This is not quite surprising, since we may naturally expect that the correlation between adjacent observations increases as the size of sampling intervals decreases. In this sense, what we observe in Figure 1 may not look entirely strange, since the KPSS test is supposed to reject the null hypothesis of stationarity or cointegration if the underlying time series is strongly persistent and has a unit longrun correlation. However, we should not interpret the results in Figure 2 as any evidence against the presence of stationarity or cointegration. The perfect correlation at high frequency here is not a consequence of the presence of a unit root in the underlying time series. Instead, it is due to the presence of strong correlation between adjacent observations, which appears in high frequency samples and has nothing to do with the persistence of the underlying time series.

Though the focus is totally different, our analysis of the KPSS test is somewhat related to that in Kwiatkowski et al. (1992) and Caner and Kilian (2001), which show that the test overly rejects the null hypothesis when the underlying process is persistent. A more systematic and comprehensive investigation of the KPSS test for highly persistent time series is given in Müller (2005). His analysis is based on a local-to-unity asymptotic framework, which assumes that the observed time series (u_i), $i = 1, \ldots, n$, follows a near unit root process $u_i = \rho_n u_{i-1} + v_i$ with $\rho_n = 1 - c/n$ for some constant $c > 0$, where (v_i) is a general stationary process satisfying the usual invariance principle. In this framework, he derives the theoretical properties of the test based on different bandwidth choices available in the literature,[6] and shows that none of the bandwidth schemes works for the KPSS test. In particular, he finds that the test has either large size distortions or very small powers, in his local-to-unity framework.

In what follows, we will develop a novel framework to effectively analyze the asymptotic properties of the KPSS test based on various bandwidth schemes, while accommodating and explaining all the features we observe in Figures 1 and 2. In the new framework, we assume that the sample of size n is collected discretely from a continuous time process at sampling interval δ over time span $[0, T]$ with $T = n\delta$. Our asymptotics rely on $\delta \to 0$ and $T \to \infty$ jointly, and whenever necessary, we let $\delta \to 0$ fast enough relative to $T \to \infty$. This is to obtain asymptotics that are more relevant for the performance of the KPSS test at

[6] Besides the above mentioned bandwidth choices for b_n, he also considers the fixed-b bandwidth that is often used to improve the finite sample performance of a test relying longrun variance estimation. See, e.g., Sun et al. (2008).

high frequency.

3. Stationarity Test at High Frequency

Throughout this section, we assume that the observed time series (u_i) is collected from an underlying continuous time process $U = (U_t)$ at sampling interval δ over time span $[0, T]$, i.e.,

$$u_i = U_{i\delta} \tag{3.1}$$

for $i = 1, \ldots, n$, with $T = n\delta$. As mentioned, we set $\delta \to 0$ and $T \to \infty$ jointly in our asymptotics. Our asymptotics will show that the KPSS test with bandwidth scheme RT or NP diverges to infinity for both stationary and nonstationary process U, whereas the KPSS test with bandwidth scheme SP has a nondegenerate limit distribution for both stationary and nonstationary process U. This is consistent with what we see in Figure 1. As a result, the test cannot be used to test for the null hypothesis of stationarity against the alternative hypothesis of nonstationarity for the underlying process U. Moreover, the subsequent development of our asymptotics will also reveal how we should implement the KPSS test at high frequency to make it valid as a test for stationarity of the underlying process U.

Below we introduce some technical assumptions, Assumptions ST and NS, respectively for stationary and nonstationary U. In what follows, we denote by $D[0, 1]$ the space of cadlag functions on $[0, 1]$ endowed with the Skorohod topology.

Assumption ST For the process $U^T = (U_t^T)$ on $[0, 1]$ defined as $U_t^T = T^{-1/2} \int_0^{tT} U_s ds$, we assume that $U^T \to U^\circ$ in $D[0, 1]$ as $T \to \infty$, where U° is a Brownian motion with variance given by $\varpi^2 = \lim_{T \to \infty} T^{-1} \mathbb{E} \left(\int_0^T U_t dt \right)^2 > 0$. Moreover, we assume $\sup_{0 \leq t \leq \infty} \mathbb{E} U_t^2 < \infty$, and $T^{-1} \int_0^T U_t^2 dt \to_p \sigma^2 > 0$.

Assumption NS For the process $U^T = (U_t^T)$ on $[0, 1]$ defined as $U_t^T = c_T^{-1} U_{Tt}$ with some normalizing sequence (c_T) such that $c_T \to \infty$ as $T \to \infty$, we assume that $U^T \to U^\circ$ in $D[0, 1]$ as $T \to \infty$, where U° is a nondegenerate stochastic process on $[0, 1]$.

Assumptions ST and NS will be used as the basic regularity conditions respectively for stationary and nonstationary U in the subsequent development of our theory. Assumption ST is satisfied for a broad class of stationary processes with mean zero and finite variance. It simply assumes three general conditions: U satisfies a mild moment condition which allows for various heterogeneities across time, the continuous time sample variance of U is consistent, and U satisfies the conventional invariance principle in continuous time, which corresponds to the same type of invariance principle for discrete stationary time series that we routinely invoke in the statistical analysis of models with unit roots and cointegration. Assumption NS just requires that under an appropriate normalization U has a well defined limit distribution, which holds for a very general class of nonstationary processes. As is well known, a wide class of discrete time series with unit root type nonstationarity satisfy this condition if we embed them into continuous time processes taking constant values between their observation intervals. The reader is referred to Park (2014) for more related discussions.

The conditions in Assumptions ST and NS are generally met for U generated as $U = \phi(V)$, where V is a diffusion and ϕ is a real-valued function defined on $\mathcal{D} \subset \mathbb{R}$. Let s and m denote the scale function and speed measure of V, respectively. The invariance principle in Assumption ST holds widely if V is a positive recurrent diffusion.[7] Indeed, Van Der Vaart and Van Zanten (2005) show that the invariance principle holds for U in this case, provided its longrun variance given by $\varpi^2 = 4m(\mathcal{D})^{-1} \int_{\mathcal{D}} \left(\int^x \phi(y) m(dy) \right)^2 s'(x) dx$ is finite. Furthermore, Kim and Park (2017) show that Assumption NS holds for general U defined from null recurrent diffusion V in natural scale with a regularly varying ϕ. If, for a general null recurrent diffusion V in natural scale, we define V^T as $V_t^T = \kappa_T^{-1} V_{Tt}$ with (κ_T) given by $\kappa_T^2 m(\kappa_T) = T$, then we have $V^T \to_d V^\circ$, where V° is a skew Bessel process in natural scale. See, e.g., Watanabe (1995) for detailed discussions on skew Bessel processes. However, it follows that

$$\frac{U_{Tt}}{\phi(\kappa_T)} = \frac{\phi(V_{Tt})}{\phi(\kappa_T)} = \frac{\phi(\kappa_T V_t^T)}{\phi(\kappa_T)} \approx \overline{\phi}(V_t^T) \to_d \overline{\phi}(V_t^\circ),$$

and Assumption NS holds for U with normalizing sequence $c_T = \phi(\kappa_T)$ and limit process $U^\circ = \overline{\phi}(V^\circ)$. The reader is referred to Kim and Park (2017) for a more rigorous development of the asymptotics discussed here.

3.1. *Continuous Time Approximation*

For the sample (u_i) drawn from a continuous time process U as in (3.1), we have

$$\frac{\delta}{n^2} \sum_{i=1}^{n} \left(\sum_{j=1}^{i} u_j \right)^2 = \frac{1}{T^2} \sum_{i=1}^{n} \delta \left(\sum_{j=1}^{i} \delta u_j \right)^2 \approx \frac{1}{T^2} \int_0^T \left(\int_0^t U_s ds \right)^2 dt \tag{3.2}$$

[7] Of course, we also need to require $m(\phi) = 0$ so that U has mean zero.

for δ small enough compared with T, and

$$\delta\omega_n^2 \approx \varpi_{n,\delta}^2 \tag{3.3}$$

with

$$\varpi_{n,\delta}^2 = \int_{|s|\leq T} K\left(\frac{s}{B_{n,\delta}}\right)\Gamma_T(s)ds, \tag{3.4}$$

where $B_{n,\delta} = b_n\delta$ and $\Gamma_T(s) = T^{-1}\int_0^T U_tU_{t-s}dt$ is the sample autocovariance function of U. Note that $K(j/b_n) = K(j\delta/b_n\delta)$ and $\gamma_n(j) = (1/n)\sum_i u_iu_{i-j} = (1/T)\sum_i \delta U_{i\delta}U_{i\delta-j\delta} \approx \Gamma_T(j\delta)$, and therefore, (3.3) follows directly from (2.2). Consequently, we may deduce that

$$\lambda_n = \frac{1}{\delta\omega_n^2}\frac{\delta}{n^2}\sum_{i=1}^n\left(\sum_{j=1}^i u_j\right)^2 \approx \frac{1}{\varpi_{n,\delta}^2}\frac{1}{T^2}\int_0^T\left(\int_0^t U_sds\right)^2dt = \Lambda_{n,\delta}, \tag{3.5}$$

where we introduce $\Lambda_{n,\delta}$ as a continuous time approximation of the KPSS test statistic λ_n. For the development of our asymptotic theory in the rest of this subsection, we introduce a set of technical conditions and formally establish the approximation in (3.5).

In what follows, we let $T(V) = \sup_{0\leq t\leq T}|V_t|$ for any continuous time process V. For a continuous time process V satisfying Assumption ST or NS, we also define

$$\Delta_{\delta,T}(V) = \sup_{0\leq s,t\leq T}\sup_{|t-s|\leq\delta}|V_t - V_s|, \tag{3.6}$$

which is interpreted as the uniform modulus of continuity of V in a δ-neighborhood over interval $[0,T]$ as $\delta \to 0$ and $T \to \infty$. We typically expect $\Delta_{\delta,T}(V) = \delta^{1/2-\varepsilon}\kappa_T$ for $\varepsilon > 0$ arbitrarily small and κ_T increasing as $T \to \infty$. If V is Brownian motion, we have $\Delta_{\delta,T}(V) = O\big(\sqrt{\delta\log(T/\delta)}\big)$, which reduces to $O\big(\sqrt{\delta\log(1/\delta)}\big)$ if $\delta \to 0$ faster relative to $T \to \infty$. This is shown in Kanaya et al. (2018). The reader is referred to Kim and Park (2017) to find an explicit rate of $\Delta_{\delta,T}(V)$ for a more general diffusion type process V. The notations introduced here will be used throughout the paper without further reference.

In this section, U appears as the only relevant continuous time process, and we will therefore simply write $\Delta_{\delta,T} = \Delta_{\delta,T}(U)$ and $T_s = T(U)$.

Assumption CA We assume that

(i) $(\Delta_{\delta,T} + \Delta_{\delta,T}^2)T \to_p 0$ under Assumption ST, and

(ii) $c_T^{-2}T_s(\Delta_{\delta,T} + \delta T_sT^{-1}) \to_p 0$ under Assumption NS,

as $\delta \to 0$ and $T \to \infty$.

The conditions in Assumption CA are satisfied as long as $\delta \to 0$ fast enough relative to $T \to \infty$. The condition in Part (i) is not very restrictive. If $\Delta_{\delta,T} = O_p(\delta^{1/2-\varepsilon})$ for $\varepsilon > 0$ arbitrarily small, then it holds as long as $\delta = O(T^{-2-\varepsilon})$ for any $\varepsilon > 0$. The condition in Part (ii) is also expected to hold widely. If U is Brownian motion, for instance, we have $c_T = \sqrt{T}$ and $T_s = O_p(\sqrt{T})$, and therefore, it is trivially satisfied as $\delta \to 0$ and $T \to \infty$.

Assumption CA is sufficient to analyze the KPSS test with RT. However, for the KPSS test with NP or SP, we need additional assumptions. Although more stringent, they are not overly restrictive and satisfied as long as $\delta \to 0$ fast enough relative to $T \to \infty$.

Assumption NP For NP with $a_n = cn^p$, we let $\delta^{1-p}T^p \to 0$ as $\delta \to 0$ and $T \to \infty$ and, under Assumption NS, we assume $c_T^{-1}T_sT^{-p/2}\delta^{p/2} \to 0$.

Assumption SP For SP, we let $U = U^c + U^d$, where $U^c = A + M$ is the continuous component with bounded variation process A and continuous martingale M, and U^d is the jump component. For the continuous component U^c, we assume $\sup_{0\leq s,t\leq T} |A_t - A_s|/|t-s| = O_p(p_T)$ and $\sup_{0\leq s,t\leq T} |[M]_t - [M]_s|/|t-s| = O_p(q_T)$, where p_T and q_T are deterministic sequences of T. For the jump component, we assume $\sum_{0\leq t\leq T} \mathbb{E}|\Delta U_t| = O(T)$ as $T \to \infty$. Moreover, under Assumption ST, we assume $\Delta_{\delta,T}(p_T\sqrt{T} + q_T) \to 0$ as $\delta \to 0$ and $T \to \infty$ and, under Assumption NS, we assume $c_T^{-2}\Delta_{\delta,T}(p_TT + q_T\sqrt{T}) \to 0$ as $\delta \to 0$ and $T \to \infty$.

The following lemma shows that the approximation in (3.5) holds under suitable regularity conditions.

Lemma 3.1. *Let Assumptions KF and CA hold. The statistic λ_n is endowed with RT, NP or SP. In particular, Assumption NP or SP holds if NP or SP is used. Then we have*

$$\lambda_n \sim_p \Lambda_{n,\delta}$$

as $\delta \to 0$ and $T \to \infty$.

3.2. Continuous Time Asymptotics

Given the asymptotic equivalence of λ_n and $\Lambda_{n,\delta}$ in Lemma 3.1 and the definition of $\varpi^2_{n,\delta}$ in (3.4), we may readily see that the asymptotic effect of sampling frequency on the KPSS test statistic λ_n can be fully analyzed by investigating the way the sampling interval δ affects $B_{n,\delta} = b_n\delta$, which is used as the bandwidth for $\varpi^2_{n,\delta}$ in (3.3). For stationary U, we may well expect that if

$$B_{n,\delta} \to_p \infty \quad \text{and} \quad B_{n,\delta}/T \to_p 0 \tag{3.7}$$

as $\delta \to 0$ and $T \to \infty$, then $\varpi_{n,\delta}^2 \to_p \varpi^2$ as $\delta \to 0$ and $T \to \infty$, where ϖ^2 is the longrun variance of U.

Following Chang et al. (2018), we say that the discrete time bandwidth b_n is *high-frequency compatible* if its corresponding continuous time bandwidth $B_{n,\delta} = b_n\delta$ satisfies conditions in (3.7). It is easy to find a discrete time bandwidth that is high-frequency compatible. For instance, for any continuous time bandwidth B_T such that $B_T \to \infty$ and $B_T/T \to 0$ as $T \to \infty$, we may just set $b_n = B_T/\delta$ for any given $\delta > 0$. Clearly, any discrete time bandwidth b_n defined in such a way is high-frequency compatible.

Unfortunately, the bandwidth b_n chosen in a discrete time setup is typically *not* high-frequency compatible. In fact, it is easy to see that RT with $b_n = cn^p$ for some $c > 0$ and $0 < p < 1$ is not high-frequency compatible, since $B_{n,\delta} = cn^p\delta = c\delta^{1-p}T^p$ and $B_{\delta,T} \to 0$ if $\delta = o(T^{-p/(1-p)})$. The asymptotics for the data-dependent procedures such as NP and SP depend upon whether the underlying process U is stationary or nonstationary. For stationary U, their asymptotics are analyzed in Chang et al. (2018) and Lu and Park (2018). In particular, they show that SP is high-frequency compatible, whereas NP is not. We summarize the asymptotics of NP and SP for both stationary and nonstationary U in the following lemma.

LEMMA 3.2. *Let Assumption CA hold. For NP satisfying Assumption NP, we have*

$$\delta^{-2r(1-p)/(2r+1)}T^{-(2rp+1)/(2r+1)}B_{n,\delta} \to_p \left(rc^{2r}\pi^2(r)\Big/(1+r)^2\iota(K^2)\right)^{1/(2r+1)}$$

as $\delta \to 0$ and $T \to \infty$, under both Assumptions ST and NS. For SP satisfying Assumption SP, (3.7) *holds under Assumption ST, and*

$$T^{-1}B_{n,\delta} \to_d \left(r(r!)^2\pi^2(r)/\iota(K^2)\right)^{1/(2r+1)} \left(\int_0^1 U_t^{\circ 2}dt\Big/\int_0^1 U_t^\circ dU_t^\circ\right)^{2r/(2r+1)}$$

as $\delta \to 0$ and $T \to \infty$, under Assumption NS.

For NP, $B_{n,\delta} \to_p 0$ if $\delta = o\big(T^{-(2pr+1)/2r(1-p)}\big)$, which holds as long as $\delta \to 0$ fast enough relative to $T \to \infty$, regardless of stationarity/nonstationarity of the underlying process U. On the other hand, SP provides a valid bandwidth, but only for stationary U. For nonstationary U, $B_{n,\delta}/T \nrightarrow_p 0$ as $\delta \to 0$ and $T \to \infty$.

THEOREM 3.3. *Let Assumptions KF and CA hold. If RT or NP satisfying Assumption NP is used, then $\lambda_n \to_p \infty$ under both Assumptions ST and NS. Moreover, if SP satisfying Assumption SP is used, then λ_n has a nondegenerate limit distribution under both Assumptions ST and NS.*

In sum, the KPSS test with any of the schemes RT, NP and SP becomes invalid at high frequency. If RT or NP is used, both the size and power of the KPSS test depend critically on the sampling interval δ, and they diverge to infinity in probability under both the null and alternative hypotheses, whenever $\delta \to 0$ fast enough compared to $T \to \infty$. Both the asymptotic size and power of the test therefore approach unity at high frequency. On the other hand, if SP is used, KPSS test has a well defined limit distribution under the null hyothesis. However, it does not diverge either but stay bounded in probability under the alternative hypothesis. The KPSS test, therefore, is inconsistent when SP is used.

3.3. Continuous Time Test and Relevant Asymptotics

In this section, we suppose that the continuous time process $U = (U_t)$ is observed continuously for $t \in [0, T]$. The continuous time KPSS test statistic is defined as

$$\Lambda_T = \frac{1}{T^2 \varpi_T^2} \int_0^T \left(\int_0^t U_s ds \right)^2 dt \tag{3.8}$$

with an estimator ϖ_T^2 for the longrun variance ϖ^2 of U, which is given by

$$\varpi_T^2 = \int_{|s| \leq T} K\left(\frac{s}{B_T} \right) \Gamma_T(s) ds, \tag{3.9}$$

where B_T is the bandwidth parameter, and $\Gamma_T(s) = T^{-1} \int_0^T U_t U_{t-s} dt$ is the sample autocovariance function of U. See Lu and Park (2018) for the longrun variance estimation of a stationary continuous time process U.

For consistency of ϖ_T^2, we require $B_T \to \infty$ and $B_T/T \to 0$ as $T \to \infty$. The optimal bandwidth that minimizes the asymptotic mean squared error of the longrun variance estimator is given by $B_T = c^*(r) T^{1/(2r+1)}$, where r is the characteristic exponent of kernel function as defined in Assumption KF, and

$$c^*(r) = \left(\frac{r \pi^2(r)}{\iota(K^2)} \Theta^2(r) \right)^{1/(2r+1)} \tag{3.10}$$

with

$$\Theta(r) = \frac{\int_{-\infty}^{\infty} |s|^r \Gamma(s) ds}{\int_{-\infty}^{\infty} \Gamma(s) ds}$$

defined correspondingly as in (2.5). Here Γ denotes the autocovariance function of U, and the parameter $\Theta(r)$ in the optimal bandwidth is unknown and should be estimated in practice.

In parallel to the discrete time bandwidth selection schemes discussed in the previous section, we consider three schemes to determine the continuous time bandwidth B_T as in Lu and Park (2018). The first scheme sets $B_T = cT^p$ for some constant $c > 0$ and $0 < p < 1$. Obviously it is the continuous time counterpart of the discrete time RT scheme, and hence it is called CRT. The second and third schemes, referred to as CNP and CSP respectively, are analogous to the discrete time NP and SP schemes. More specifically, CNP sets $B_T = c_T^*(r)T^{1/(2r+1)}$, where $c_T^*(r)$ is the estimate for $c^*(r)$ in (3.10) obtained with $\Theta(r)$ replaced by its nonparametric estimate

$$\Theta_T(r) = \frac{\int_{|s|\leq A_T} |s|^r \Gamma_T(s)ds}{\int_{|s|\leq A_T} \Gamma_T(s)ds} \tag{3.11}$$

with $A_T = cT^p$ for some arbitrarily chosen constants $c > 0$ and $0 < p < 1$. Similarly as SP, CSP assumes that U follows a semiparametric continuous time model

$$dU_t = -\kappa U_{t-}dt + \upsilon(U_{t-})dW_t + \omega(U_{t-})dJ_t, \tag{3.12}$$

where υ and ω are unknown volatility functions respectively for the diffusive and jump parts of U, W is standard Brownian motion and J is the Lévy jump process defined by $dJ_t = \int_{\mathbb{R}} xN(dt, dx)$ from the Poisson random measure N with the corresponding Lévy measure ν such that $\int_{\mathbb{R}} x\nu(dx) = 0$. Under this specification, we have $\Theta(r) = r!/\kappa^r$ if U is stationary with finite variance, as shown in Proposition 3.2 of Lu and Park (2018). Hence we may obtain the estimate $\Theta_T(r)$ of $\Theta(r)$ from an estimate κ_T of κ, and use it to find an estimate for the optimal bandwidth as $B_T = c_T^*(r)T^{1/(2r+1)}$, where $c_T^*(r)$ is given as in (3.10) with $\Theta(r)$ replaced by $\Theta_T(r) = r!/\kappa_T^r$. For an estimate κ_T of κ, we will mainly consider the continuous time least squares estimator $\kappa_T = -\int_0^T U_t dU_t \Big/ \int_0^T U_t^2 dt$, though there are other possibilities.[8]

The asymptotics of Λ_T in (3.8) will be derived under the null and alternative hypotheses, respectively, under Assumptions ST and NS. The null asymptotics are straightforward, because if U is stationary and satisfies Assumption ST, ϖ_T^2 is a consistent estimate for ϖ^2 with a proper bandwidth choice and we have

$$\frac{1}{T^2}\int_0^T \left(\int_0^t U_s ds\right)^2 dt = \int_0^1 \left(\frac{1}{\sqrt{T}}\int_0^{tT} U_s ds\right)^2 dt \to_d \varpi^2 \int_0^1 W_t^2 dt$$

[8]We find that the choice of an estimator κ_T is unimportant. It does not affect our simulation result in any significant manner, as well as asymptotic theory.

as $T \to \infty$, from which it follows immediately that

$$\Lambda_T \to_d \int_0^1 W_t^2 dt \tag{3.13}$$

as $T \to \infty$.

The asymptotics of Λ_T under the alternative hypothesis are much less trivial, especially when a data dependent bandwidth is used. In Lemma 3.4, we first derive the asymptotics of B_T for CNP and CSP under the alternative hypothesis.

Lemma 3.4. *Under Assumption NS, we have*

(a) $T^{-(2rp+1)/(2r+1)} B_T \to_p \left(rc^{2r}\pi^2(r)/(1+r)^2 \iota(K^2)\right)^{1/(2r+1)}$, *if CNP is used, and*

(b) $T^{-1} B_T \to_d \left(r(r!)^2 \pi^2(r)/\iota(K^2)\right)^{1/(2r+1)} \left(\int_0^1 U_t^{\circ 2} dt / \int_0^1 U_t^\circ dU_t^\circ \right)^{2r/(2r+1)}$, *if CSP is used,*

as $T \to \infty$, *where* U° *is the limit process of* U *defined in Assumption NS.*

Lemma 3.4 shows that, under the alternative hypothesis, CNP yields a bandwidth such that $B_T \prec_p T$, whereas we have $B_T \sim_p T$ for the bandwidth given by CSP. Note in particular that the bandwidth obtained from CRT or CNP diverges at a slower rate than that from CSP under the alternative hypothesis.

In what follows, we let $\Gamma^\circ(s) = \int_0^1 U_t^\circ U_{t-s}^\circ dt$ for $-1 \leq s \leq 1$, where U° is the limit process of U defined in Assumption NS. Moreover, whenever we assume $B_T/T \to_d B^\circ$ for some $B^\circ \neq 0$ a.s., it is meant to hold jointly with $U^T \to_d U^\circ$ implied by Assumption NS. The asymptotics of ϖ_T^2 under the alternative hypothesis are presented in the following lemma.

Lemma 3.5. *Let Assumption KF hold. Under Assumption NS, we have*

(a) $c_T^{-2} B_T^{-1} \varpi_T^2 \to_d \iota(K) \int_0^1 U_t^{\circ 2} dt$, *if* $B_T/T \to_p 0$, *and*

(b) $c_T^{-2} T^{-1} \varpi_T^2 \to_d \int_{-1}^1 K(s/B^\circ)\Gamma^\circ(s) ds$, *if* $B_T/T \to_d B^\circ \neq 0$ *a.s.,*

as $T \to \infty$.

The asymptotics of Λ_T under the alternative hypothesis now follow immediately from Lemma 3.5, and they are presented in the following proposition.

Proposition 3.6. *Let Assumption KF hold. Under Assumption NS, we have*

(a) $(B_T/T)\Lambda_T \to_d \int_0^1 \left(\int_0^t U_s^\circ ds \right)^2 dt / \iota(K) \int_0^1 U_t^{\circ 2} dt$, *if* $B_T/T \to_p 0$, *and*

(b) $\Lambda_T \to_d \int_0^1 \left(\int_0^t U_s^\circ ds\right)^2 dt / \int_{-1}^1 K(s/B^\circ)\Gamma^\circ(s)ds$, *if* $B_T/T \to_d B^\circ \neq 0$ *a.s.*,

as $T \to \infty$.

The asymptotics in Part (a) of Proposition 3.6 are applicable to the continuous time KPSS test employed with the bandwidth given by CRT or CNP. Therefore, under these two bandwidth selection schemes, we have $\Lambda_T \sim_p T/B_T$, which implies $\Lambda_T \to_p \infty$ as $T \to \infty$. Therefore, the test is consistent under these two schemes. Unfortunately, however, the use of CSP invalidates the continuous time KPSS test. In fact, the asymptotics in Part (b) of Proposition 3.6 show that, if CSP is used, the test statistic Λ_T is stochastically bounded and the test becomes inconsistent. As can be seen clearly from Part (b) of Lemma 3.4 and 3.5, the bandwidth under CSP increases too fast and makes the resulting longrun variance estimate explode as $T \to \infty$, when the underlying process is nonstationary.

4. Residual Based KPSS Test at High Frequency

In this section, we consider the KPSS tests based on the fitted residuals (u_{ni}) defined in (2.3) and (2.4), which are commonly used in practical applications for the tests of stationarity and cointegration respectively. The asymptotics of these tests can easily be derived from our asymptotics developed in the previous sections.

4.1. Asymptotic Test

The RB-KPSS test defined from the fitted residuals in (2.3) at high frequency can be used to test for stationarity of an underlying continuous time process possibly with nonzero mean. On the other hand, the RB-KPSS test based on the fitted residuals in (2.4) at high frequency can be used to test for the presence of cointegration between two continuous time processes $Y = (Y_t)$ and $X = (X_t)$, if (y_i) and (x_i) defined as $y_i = Y_{i\delta}$ and $x_i = X_{i\delta}$ are observed at high frequency for $i = 1, \ldots, n$. To analyze the RB-KPSS test of cointegration, we introduce Assumptions CI and NC, which are assumed to hold respectively under the null hypothesis of cointegration and the alternative hypothesis of no cointegration.

Assumption CI For the processes $Y^T = (Y_t^T)$ and $X^T = (X_t^T)$ defined on $[0, 1]$ respectively as $Y_t^T = c_T^{-1} Y_{Tt}$ and $X_t^T = c_T^{-1} X_{Tt}$ with some normalizing sequence (c_T) such that $c_T \to \infty$ as $T \to \infty$, we assume that $Y^T \to_d Y^\circ$ and $X^T \to_d X^\circ$ jointly as $T \to \infty$, where both Y° and X° are non-degenerate stochastic processes on $[0, 1]$. Moreover, we assume that there exist some constants α and β such that the continuous time process $U = (U_t)$ defined as $U_t = Y_t - \alpha - \beta X_t$ satisfies Assumption ST.

Assumption NC For the process $X^T = (X_t^T)$ defined on $[0,1]$ as $X_t^T = d_T^{-1} X_{Tt}$ with some normalizing sequence (d_T) such that $d_T \to \infty$, we assume that $X^T \to_d X^\circ$ as $T \to \infty$. Moreover, for any values of α and β, the continuous time process $U = (U_t)$ defined as $U_t = Y_t - \alpha - \beta X_t$ satisfies Assumption NS.

The definition of cointegration between continuous time processes we use in Assumptin CI is comparable to that of cointegration between discrete time series. However, our formulation of cointegration here is more general in that we allow for arbitrary normalization sequences and for general continuous time processes not necessarily converging to Brownian motion in the limit. In our setup, it is meaningless to consider continuous time processes requiring normalization sequences of different orders, since cointegrating relationship is not meaningfully defined in such a case.

To facilitate our analysis of the RB-KPSS test relying on the test statistic λ_n obtained from the fitted residuals (u_{ni}), we write u_{ni} in (2.3) and (2.4) commonly as

$$u_{ni} = v_i - \bar{v}_n$$

with $\bar{v}_n = n^{-1} \sum_{j=1}^n v_j$ by letting $v_i = u_i$ and $u_i - \left(\sum_{j=1}^n u_j (x_j - \bar{x}_n) / \sum_{j=1}^n (x_j - \bar{x}_n)^2 \right) x_i$ for (2.3) and (2.4) respectively. Correspondingly, we define a continuous time process V as

$$V_t = U_t \quad \text{and} \quad U_t - \left(\int_0^T (X_s - \bar{X}_T) U_s ds \bigg/ \int_0^T (X_s - \bar{X}_T)^2 ds \right) X_t \tag{4.1}$$

with $\bar{X}_T = T^{-1} \int_0^T X_s ds$ respectively. We will show that λ_n is asymptotically equivalent to Λ_T, the corresponding continuous time RB-KPSS test statistic defined as in (3.8) with U replaced by $V - \bar{V}_T$, where $\bar{V}_T = T^{-1} \int_0^T V_t dt$.

Below we introduce the regularity conditions required for the continuous time approximation of the RB-KPSS test statistic λ_n.

Assumption CA′ (i) If Assumption CI holds, then we assume that Assumption CA (i) holds with $\Delta_{\delta,T}$ replaced by $\Delta_{\delta,T}(U) + T^{-1/2} c_T^{-1} \Delta_{\delta,T}(X)$. (ii) If Assumption NC holds, then we assume that Assumption CA (ii) holds with $\Delta_{\delta,T}$ replaced by $\Delta_{\delta,T}(U) + c_T d_T^{-1} \Delta_{\delta,T}(X)$, and T_s replaced by $T(U) + c_T d_T^{-1} T(X)$.

The following lemma establishes the asymptotic equivalence between the discrete time RB-KPSS test statistic λ_n and its continuous time counterpart Λ_T. Here we only consider the test statistic defined with high-frequency compatible bandwidths. It is clear from our previous analysis in Section 3.2 that the test becomes invalid if any of high-frequency incompatible bandwidths is used.

LEMMA 4.1. *Let Assumption KF hold. We have*

$$\lambda_n \sim_p \Lambda_T$$

as $\delta \to 0$ *and* $T \to \infty$*, under Assumptions CA and CA′ for the tests of stationarity and cointegration, respectively.*

To establish the asymptotic equivalence of λ_n and Λ_T, Assumption CA is sufficient for the stationarity test based on the residuals in (2.3), whereas Assumption CA′ is required for the cointegration test based on the residuals in (2.4). The asymptotic null distributions of the RB-KPSS tests for the stationarity and cointegration tests, which can be easily deduced from Lemma 4.1, are provided in the following theorem.

THEOREM 4.2. *Let Assumption KF and CA′ hold. We have*

$$\lambda_n \to_d \int_0^1 \left(\int_0^t dW_s^\circ \right)^2 dt$$

as $\delta \to 0$ *and* $T \to \infty$*, where* $dW_s^\circ = dW_s - W_1 ds$ *under Assumption ST, and* $dW_s^\circ = (dW_s - W_1 ds) - \left[\int_0^1 (X_t^\circ - \bar{X}_1^\circ) dW_t / \int_0^1 (X_t^\circ - \bar{X}_1^\circ)^2 dt \right] (X_s^\circ - \bar{X}_1^\circ) ds$ *with* $\bar{X}_1^\circ = \int_0^1 X_t^\circ dt$ *under Assumption CI.*

The RB-KPSS test for stationarity has the limit null distribution given as a functional of the standard Brownian bridge, exactly as in the discrete time setup, and therefore, we may just use the conventional critical values. The limit null distribution of the RB-KPSS test for cointegration generalizes the one obtained in the discrete time setup. In the special case where X^* is given as another Brownian motion, our limit distribution reduces to the one in Shin (1994) obtained in the discrete time setup and his critical values become applicable. In our continuous time setup, X° is given as a general limit process and the asymptotic critical values of the RB-KPSS test for cointegration become heavily model-dependent. To deal with this problem, we propose to use a subsample test relying on a modified version of the RB-KPSS test. This will be introduced in the next section.

To derive the asymptotics of the RB-KPSS tests under the alternative hypotheses of nonstationarity and no cointegration, we let $V^T = (V_t^T)$ as $V_t^T = c_T^{-1} V_{Tt}$ for $t \in [0,1]$, where V is defined in (4.1). Under the alternative hypotheses of nonstationarity and no cointegration, we have $V^T \to V^\circ$ in $D[0,1]$ as $T \to \infty$, where $V^\circ = U^\circ$ with U° defined in Assumption NS for the stationarity test, and $V^\circ = U^\circ - \big[\int_0^1 (X_t^\circ - \bar{X}_1^\circ) U_t^\circ dt / \int_0^1 (X_t^\circ - \bar{X}_1^\circ)^2 dt\big] X^\circ$ with X° and U° defined in Assumption NC for the cointegration test. In what follows, we let $\bar{V}_1^\circ = \int_0^1 V_t^\circ dt$. We first present the asymptotics of CNP and CSP under the alternative hypotheses of nonstationarity and no cointegration.

LEMMA 4.3. *Let Assumption KF hold. Under Assumption NS with Assumption CA or Assumption NC with Assumption CA′, we have*

(a) $T^{-(2rp+1)/(2r+1)} B_T \to_p \big(rc^{2r}\pi^2(r)/(1+r)^2 \iota(K^2)\big)^{1/(2r+1)}$, *if CNP is used*

(b) $T^{-1} B_T \to_d \big(r(r!)\pi^2(r)/\iota(K^2)\big)^{1/(2r+1)} \Big(\int_0^1 (V_t^\circ - \bar{V}_1^\circ)^2 dt / \int_0^1 (V_t^\circ - \bar{V}_1^\circ) dV_t^\circ\Big)^{2r/(2r+1)}$, *if CSP is used*

as $\delta \to 0$ *and* $T \to \infty$.

Lemma 4.3 is directly comparable to Lemma 3.4. For the RB-KPSS tests, we also have $B_T \prec_p T$ for CNP, while $B_T \sim_p T$ for CSP, under the alternative hypotheses. The asymtotics of the RB-KPSS tests are provided in the following theorem.

THEOREM 4.4. *Let Assumption KF hold. Under Assumption NS with Assumption CA or Assumption NC with Assumption CA′, we have*

(a) $(B_T/T)\lambda_n \to_d \int_0^1 \big(\int_0^t (V_s^\circ - \bar{V}_1^\circ) ds\big)^2 dt / \iota(K) \int_0^1 (V_t^\circ - \bar{V}_1^\circ)^2 dt$, *if* $B_T/T \to 0$

(b) $\lambda_n \to_d \int_0^1 \big(\int_0^t (V_s^\circ - \bar{V}_1^\circ) ds\big)^2 dt / \int_{-1}^1 K(s/B^\circ)\Gamma^\circ(s) ds$, *if* $B_T/T \to B^\circ \neq 0$ *a.s.*

as $\delta \to 0$ *and* $T \to \infty$, *where* $\Gamma^\circ(s) = \int_0^1 (V_t^\circ - \bar{V}_1^\circ)(V_{t-s}^\circ - \bar{V}_1^\circ) dt$.

Theorem 4.4 implies in particular that the RB-KPSS tests with CRT and CNP are consistent, while the tests become inconsistent if CSP is used.

4.2. *Subsample Test*

As discussed, the RB-KPSS test for cointegration has limit distribution that is model-dependent in our general continuous time setup. To resolve this problem, we consider a subsample test. To make our subsample test high-frequency compatible, we set subsample size as $m = S_T/\delta$, where S_T is given by $S_T = cT^q$ with some constant $c > 0$ and $0 < q < 1$. Our subsample test is based on a modified version λ_n^* of the KPSS test statistic λ_n, which is defined as

$$\lambda_n^* = \frac{\delta}{n^2} \sum_{i=1}^n \left(\sum_{j=1}^i u_{nj} \right)^2. \tag{4.2}$$

The modified KPSS test, therefore, does not have a longrun variance estimate in the denominator. As shown, the presence of longrun variance estimate in λ_n results in some significant asymptotic power loss,[9] and therefore, we purposely avoid estimating the longrun variance in our subsample test.

The following corollary presents the asymptotics of the modified RB-KPSS test based on λ_n^* defined in (4.2) under the null and alternative hypotheses.

Corollary 4.5. *We have*

(a) $\lambda_n^* \to_d \int_0^1 \left(\int_0^t d\tilde{W}_s^\circ \right)^2 dt$ *under Assumption ST with Assumption CA or Assumption CI with Assumption CA′, where* $d\tilde{W}_s^\circ = \varpi dW_s^\circ$ *for* dW_s° *defined in Theorem 4.2, and*

(b) $c_T^{-2} T^{-1} \lambda_n^* \to_d \int_0^1 \left(\int_0^t (V_s^\circ - \bar{V}_1^\circ) ds \right)^2 dt$ *under Assumption NS with Assumption CA or Assumption NC with Assumption CA′,*

as $\delta \to 0$ *and* $T \to \infty$.

Given the asymptotics in Corollary 4.5, the validity of the subsample test based on λ_n^* can be readily established. In particular, when subsample size is $m = S_T/\delta$ with $S_T = cT^q$ for $c > 0$ and $0 < q < 1$, the critical value we obtain from subsampling λ_n^* diverges at rate $(c_T^2 T)^q$ as $T \to \infty$, under the alternative hypothesis of nonstationarity or no cointegration. Consequently, the power of our subsample test increases at rate $(c_T^2 T)^{1-q}$ as $T \to \infty$.

In terms of asymptotic discriminatory power, the subsample test relying on the modified statistic λ_n^* is unambiguously preferred to the subsample test using the original statistic λ_n that includes a longrun variance estimate. For the subsample test based on λ_n, the asymptotic discriminatory power depends upon the bandwidth selection for longrun variance estimation. In fact, it follows immediately from our asymptotic theory that the subsample test based on λ_n becomes inconsistent if CSP is used, and its asymptotic power increases at rates $T^{(1-p)(1-q)}$ and $T^{[2r(1-p)/(2r+1)](1-q)}$ as $T \to \infty$ respectively if CRT and CNP are used.[10] Clearly, the subsample test using λ_n^* has a greater asymptotic discriminatory power than the subsample test relying on λ_n with any choice of the bandwidth selection procedures considered in the paper. Note that $c_T \to \infty$ as $T \to \infty$, and therefore, $T^{1-q} = o((c_T^2 T)^{1-q})$ as $T \to \infty$.

[9] This is the reason why our subsample approach here is also useful for the KPSS test for stationarity, as well as the KPSS test for cointegration.

[10] Recall that for CRT we set bandwidth $b_n = cT^p/\delta$ in evaluating test statistic λ_n, and for CNP we set bandwidth $a_n = cT^p/\delta$ in estimating parameter $\Theta(r)$ nonparametrically, with some constant $c > 0$.

5. Simulations

In our simulations, we consider the Ornstein-Uhlenbeck (OU) process given by

$$dU_t = \kappa(\mu - U_t)dt + \sigma dW_t, \tag{5.1}$$

where $-\infty < \mu < \infty, \kappa \geq 0$ and $\sigma > 0$ are parameters and W is standard Brownian motion. The parameter κ, called the mean reversion parameter, determines the persistency of U. If $\kappa > 0$, U becomes stationary with invariant distribution given by $\mathbb{N}(\mu, \sigma^2/2\kappa)$. On the other hand, if $\kappa = 0$, U becomes Brownian motion (BM), and therefore, nonstationary. For the RB-KPSS test of stationarity, U is specified under the null hypothesis as the OU process with two sets of parameter values, $(\kappa, \sigma^2) = (5, 0.006^2)$ and $(0.2, 0.006^2/25)$. The OU processes with these two sets of parameter values are referred to as OU-T and OU-P, where T and P signify transitory and persistent models, respectively. Note that the value of the mean reversion parameter κ for OU-T is substantially larger than OU-P, which means that the former (the latter) is more transitory (persistent) than the latter (the former).[11] Under the alternative hypothesis, U is specified as BM with $\kappa = 0$.

For the RB-KPSS test of cointegration, we also consider

$$X_t = \varsigma V_t, \tag{5.2}$$

where $\varsigma > 0$ is a parameter and V is standard BM, and let $Y = X + U$. Under the null hypothesis, we assume that U is generated as OU-T and OU-P, introduced above, and that $d[V, W]_t = \rho dt$, where $-1 \leq \rho \leq 1$ is the correlation coefficient between V and W, to allow for dependence between V and W, where W is standard Brownian motion introduced in (5.1). We set $\varsigma = 0.099$ and $\rho = -0.034$.[12] Under the alternative hypothesis, we simply set $\kappa = 0$ and $\rho = 0$, so that U and X become independent BMs.

The exact transitions are used to generate daily samples of OU processes and BMs for $T = 10, 30$ and 50 years of sample span, and samples are collected at intervals ranging from $\delta = 1/252$ to $1/4$ corresponding to daily and quarterly frequencies. We draw the initial values of stationary OU processes from their invariant distributions. The reported rejection

[11] The values of κ and σ^2 in OU-T are obtained from the fitted OU process of the 1-month forward premium of US/UK exchange rates, and the value of κ in OU-P is comparable to the estimate of the OU process fitted by the US 3-month T-bill rates. See the notes in Figure 1 for the details of the data used here.

[12] The values of ς and ρ are set to be identical to those we obtain from the daily US/UK exchange rates and their 1-month forward premium respectively. To estimate ρ, we observe from (5.1) and (5.2) that $d[U, X]_t = \sigma\varsigma d[W, V]_t = \sigma\varsigma\rho dt$, or, $T^{-1}[U, X]_T = \sigma\varsigma\rho$. Therefore, estimate of ρ can be obtained as $\hat{\rho} = T^{-1}\sum_{i=1}^{n}(u_i - u_{i-1})(x_i - x_{i-1})/(\hat{\sigma}\hat{\varsigma})$.

probabilities are computed based on 5000 simulation iterations. We use Parzen kernel[13] with six bandwidth selection schemes RT, NP, SP, CRT, CNP and CSP. For three conventional discrete time schemes RT, NP and SP, we follow the literature to choose the relevant constants.[14] For continuous time schemes CRT, CNP and CSP, we let $b_n = B_T/\delta$. We set $B_T = cT^{1/4}$ with $c = 0.5886$ for CRT.[15] For CNP and CSP, we let $B_T = 2.6614\Theta_T(2)^{2/5}T^{1/5}$. We use the first step bandwidth $A_T = 0.5886T^{1/4}$ for CNP, and set $\Theta_T(2) = 4/\kappa_T^2$ for CSP.

5.1. *Frequency Dependence of KPSS Test*

To investigate the frequency dependence of the RB-KPSS tests for stationarity and cointegration, we compute their rejection probabilities for variant values of sampling interval δ while fixing sample span at $T = 50$.[16] Our simulation results are illustrated in Figures 3 and 4, respectively for the test of stationarity and the test of cointegration. Overall, the tests with conventional discrete time schemes RT and NP or their continuous time versions CRT and CNP behave quite distinctively from those with conventional discrete time scheme SP and its continuous time version CSP. Recall that the former are consistent, while the latter are inconsistent. For the consistent tests, massive over-rejections of the null hypotheses of stationarity and cointegration are observed when we use highly persistent OU-P in our simulations. As discussed earlier in Section 2, this is a well known problem already observed and analyzed in discrete time framework by many authors. On the contrary, the use of persistent OU-P does not affect the inconsistent tests in any noticeable manner. Below we will focus on our simulations with OU-T, since our main purpose here is to study the frequency dependence, not to evaluate the overall performance of the tests.

Figures 3 and 4 demonstrate that our asymptotics are very relevant and useful. For

[13] It is well recognized that the choice of kernel function plays a secondary role in determining the properties of the test, hence we only consider one widely used kernel function, Parzen kernel, as an illustration and we expect the results mainly hold if any other proper kernel function is used. The character exponent $r = 2$ for Parzen kernel.

[14] We set $b_n = 12(n/100)^{1/4}$ for RT (Kwiatkowski et al. (1992)), $b_n = 2.6614\theta_n(2)^{2/5}n^{1/5}$ for NP, where $\theta_n(2)$ is calculated with $a_n = 4(n/100)^{4/25}$ (Newey and West (1994)), and $b_n = 2.6614\theta_n(2)^{2/5}n^{1/5}$ for SP, where $\theta_n(2) = 2\hat{\rho}/(1-\hat{\rho})^2$ with $\hat{\rho}$ being the estimated AR coefficient in the AR(1) regression using the discrete sample (Andrews (1991)).

[15] As in RT, there is no solid rule to set the constant in CRT, so we choose it to be comparable to that in RT. In particular, we set c such that $B_T = cT^{1/4} = b_n\delta$ where $b_n = 3.7947n^{1/4}$ under RT, for $\delta = 1/12$ corresponding to monthly frequency used commonly in discrete studies, and $n = 1000$ which is usually the largest sample size adopted in discrete time simulation studies (e.g., Newey and West (1994)). Therefore, we set $c = \frac{1/12\times 3.7947\times 1000^{1/4}}{(1/12\times 1000)^{1/4}} = 0.5886$.

[16] Our asymptotics require $T \to \infty$ as well as $\delta \to 0$. In our simulations here, we let T be fixed to focus on how the rejection probabilities change as $\delta \to 0$. Of course, it is possible to vary T simultaneously with δ as in our asymptotics. Though we do not report the details, our simulation results in this setup are qualitatively identical to those we present here with T fixed.

the tests of stationarity and cointegration with RT and NP, the rejection probabilities especially under the null hypotheses of stationarity and cointegration increase as δ shrinks, very sharply as δ goes beyond the monthly frequency and approaches the daily frequency. Only the tests with SP, which are inconsistent, show no such tendency. On the other hand, the rejection probabilities of the tests with CRT and CNP do not show any frequency dependence, staying more or less constant at all frequencies. This is true under both the null and alternative hypotheses. Again, the tests with SP, as well CSP, yield very stable rejection probabilities across all frequencies. They are more stable under the null hypothesis than under the alternative hypothesis. Though they are shown to have some powers, they are trivial since they do not increase with T. This is shown in the next section.

Given the frequency invariance of the test performance with continuous time bandwidth schemes, there seems no compelling reason why we should use high frequency observations. However, we find that the tests using higher frequency observations are generally much more robust in the sense that the possibility of making contradictory rejection decisions is much lower when we change the sampling interval slightly. To demonstrate this, we count, at each given sampling frequency, the number of contradictory rejection decisions resulting from any change in sampling frequency by one day up to two days and five days. The counts of contradictory rejection decisions are reported as the relative percentages to the total number of simulation iterations. Figure 5 shows a clear upward trend in the contradiction rate as δ increases, which implies that if we use low frequency observations, we are more likely to face contradictory results when a small and insignificant change in sampling frequency is made.

5.2. Size and Power of Asymptotic and Subsample Tests

Table 1 presents the rejection probabilities of the RB-KPSS test for stationarity with CRT, CNP and CSP. Our simulation results reported in Table 1 are consistent with the asymptotic theory developed in the paper. In particular, they show that the test with CSP yields no nontrivial asymptotic power. Its power does not increase with T, though it appears to have some power in finite samples. The test has the largest power if CRT is used. The test with CNP does not perform as well as the test with CRT. However, it has power that increases as T gets large, in contrast to the test with CSP.

In Table 2, we report the rejection probabilities of the RB-KPSS test for cointegration with CRT, CNP and CSP, and the subsample test of cointegration based on the modified RB-KPSS test statistic, which are referred to respectively as the asymptotic test and the modified subsample (MS) test for short. For the MS test, we set subsample size $S_T = T^{1/2}$ and subsamples are drawn at 1-month intervals. Our simulation results for the asymptotic

Table 1. Rejection Probabilities of Stationarity Tests

	OU-T			OU-P			BM		
	$T=10$	$T=30$	$T=50$	$T=10$	$T=30$	$T=50$	$T=10$	$T=30$	$T=50$
CRT	0.071	0.072	0.065	0.610	0.741	0.755	0.711	0.901	0.955
CNP	0.053	0.055	0.050	0.249	0.334	0.317	0.393	0.621	0.709
CSP	0.047	0.060	0.055	0.212	0.038	0.010	0.359	0.350	0.363

Notes: Reported are the rejection probabilities of asymptotic stationarity tests with CRT, CNP and CSP schemes under the null (OU-T and OU-P) and the alternative (BM) hypotheses, using daily observations for variant sample spans.

Table 2. Rejection Probabilities of Cointegration Tests

	OU-T Error			OU-P Error			BM Error		
	$T=10$	$T=30$	$T=50$	$T=10$	$T=30$	$T=50$	$T=10$	$T=30$	$T=50$
CRT	0.084	0.067	0.073	0.498	0.691	0.734	0.587	0.865	0.940
CNP	0.139	0.069	0.063	0.273	0.302	0.285	0.356	0.516	0.606
CSP	0.072	0.061	0.063	0.308	0.172	0.118	0.409	0.424	0.422
MS	0.058	0.040	0.037	0.341	0.422	0.414	0.436	0.676	0.778

Notes: Reported are the rejection probabilities of asymptotic and modified subsample (MS) cointegration tests under the null (OU-T and OU-P Errors) and the alternative (BM Errors) hypotheses, using daily observations for variant sample spans. The asymptotic tests are implemented with CRT, CNP and CSP schemes. For the modified subsample test (MS), we set subsample size $S_T = T^{1/2}$ and subsamples are drawn with 1-month intervals.

test here are also largely consistent with our asymptotic theory. The test with CSP has no nontrivial power, and the test with CRT performs best. The performance with CNP is not impressive, but at least its power increases steadily as T gets large. The MS test generally provides more proper size than the asymptotic test, unless the error term becomes highly persistent. The power performance of the MS test improves as T increases, and it is better than that of the asymptotic test with CNP or CSP, but worse than that with CRT. In terms of both size and power, the MS test appears to perform comparably with the asymptotic test with CRT. Both the MS test and the asymptotic test are valid in our simple simulation setup. Note, however, that only the MS test is applicable for the test of cointegration in the more general context considered in the paper.

6. Asymptotics for Variants of KPSS Tests

Before closing the paper, we briefly show in this section that the frequency dependence problem shown in Figure 1 and formally analyzed in Sections 3.1 and 3.2 is applicable not only for the KPSS test defined in (2.1), but also for its variants defined similarly. In particular, we use our continuous time framework to examine the high-frequency behaviors of the modified R/S statistic proposed by Lo (1991) and the rescaled variance test developed in Giraitis et al. (2003), both of which are proposed for the test of long memory.

For a given time series $(u_i), i = 1, \ldots, n$ with its longrun variance estimate ω_n^2 defined in (2.2), the modified R/S statistic can be written as[17]

$$\lambda_n^{\mathrm{RS}} = \frac{1}{\sqrt{n}\omega_n} \left(\max_{1 \leq i \leq n} \sum_{j=1}^{i} u_j - \min_{1 \leq i \leq n} \sum_{j=1}^{i} u_j \right),$$

and the rescaled variance test statistic, or V/S statistic, as

$$\lambda_n^{\mathrm{VS}} = \frac{1}{n^2 \omega_n^2} \left[\sum_{i=1}^{n} \left(\sum_{j=1}^{i} u_j \right)^2 - \frac{1}{n} \left(\sum_{i=1}^{n} \sum_{j=1}^{i} u_j \right)^2 \right].$$

Like the KPSS test statistic λ_n in (2.1), modified R/S statistic λ_n^{RS} and V/S statistic λ_n^{VS} are also defined from the partial sum (s_i) of (u_i) given by $s_i = \sum_{j=1}^{i} u_j$ for $i = 1, \ldots, n$, with a longrun variance estimate ω_n^2 or its square root ω_n in the denominator. Note that λ_n^{RS} is based on the range of (s_i), and λ_n^{VS} essentially looks at the sample variance of (s_i).

Following our analysis in Section 3.1, we first establish the asymptotic equivalence of λ_n^{RS} and λ_n^{VS} to their continuous time counterparts defined as

$$\begin{aligned}
\Lambda_{n,\delta}^{\mathrm{RS}} &= \frac{1}{\varpi_{n,\delta}} \left(\max_{0 \leq t \leq 1} \int_0^{tT} U_s ds - \min_{0 \leq t \leq 1} \int_0^{tT} U_s ds \right) \\
\Lambda_{n,\delta}^{\mathrm{VS}} &= \frac{1}{\varpi_{n,\delta}^2} \left[\frac{1}{T^2} \int_0^T \left(\int_0^t U_s ds \right)^2 dt - \frac{1}{T^3} \left(\int_0^T \int_0^t U_s ds dt \right)^2 \right].
\end{aligned}$$

The asymptotic equivalence is shown in the following corollary to Lemma 3.1. As before, we consider RT, NP and SP.

Corollary 6.1. *Let Assumptions KF and CA hold, and assume $T_s T^{-1} \delta \prec_p \Delta_{\delta,T}$. Fur-*

[17] λ_n^{RS} is actually $n^{-1/2}$ multiple of the original modified R/S statistic in Lo (1991).

thermore, let Assumption NP or SP hold if NP or SP is used. Then we have

$$\sqrt{T}\lambda_n^{\mathrm{RS}} \sim_p \Lambda_{n,\delta}^{\mathrm{RS}} \quad \textit{and} \quad \lambda_n^{\mathrm{VS}} \sim_p \Lambda_{n,\delta}^{\mathrm{VS}}$$

as $\delta \to 0$ *and* $T \to \infty$*.*

Now we may easily deduce the asymptotics of λ_n^{RS} and λ_n^{VS}, which we present as a corollary to Theorem 3.3 below.

COROLLARY 6.2. *Let Assumptions KF and CA hold. If RT or NP satisfying Assumption NP is used, then* $\lambda_n^{\mathrm{RS}} \to_p \infty$ *and* $\lambda_n^{\mathrm{VS}} \to_p \infty$ *under both Assumptions ST and NS. Moreover, if SP satisfying Assumption SP is used, then* λ_n^{RS} *and* λ_n^{VS} *have nondegenerate limit distributions under both Assumptions ST and NS.*

Therefore, the modified R/S test and the rescaled variance test with the usual bandwidth choices are also invalid if the discrete samples are obtained at high frequency. We expect over-rejection of the null hypothesis if RT or NP is used, and the lack of power if SP is used.

7. Conclusion

In this paper, we consider testing for stationarity using high frequency observations. In particular, we study the asymptotic properties of the KPSS test as a stationarity test and a residual based cointegration test using a continuous time framework. We find that if high frequency observations are used, the test employing any conventional bandwidth selection scheme does not have discriminatory power between stationary and nonstationary processes. We propose using a continuous time bandwidth selection approach, CRT or CNP. The test using either scheme is consistent and not sensitive to sampling frequency. Moreover, our simulation results indicate that the test result is more stable if higher frequency observations are used. In this paper, we also show that the KPSS test, as a stationarity test, has asymptotic null distribution identical to that of its counterpart in the usual discrete time framework. Therefore, the critical values tabulated in Kwiatkowski et al. (1992) are applicable. However, the KPSS test, as a residual based cointegration test, has asymptotic distribution dependent upon various nuisance parameters, which makes asymptotic test infeasible. We propose a modified subsampling test for testing cointegration. The test can also be used as a stationarity test. The simulation results are generally consistent with our asymptotic theories.

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Appendices

A. Useful Lemmas and Their Proofs

Lemma A.1. *Let Assumption KF hold. For $B_{n,\delta} = \delta b_n$ satisfying $\delta \prec_p B_{n,\delta} \prec_p T$, we have*
(a) $\sum_{|j|\leq n-1} \int_{j\delta}^{(j+1)\delta} K(s/B_{n,\delta})ds = O(B_{n,\delta})$, *and*
(b) $\sum_{|j|\leq n-1} \int_{j\delta}^{(j+1)\delta} [K(s/B_{n,\delta}) - K(j\delta/B_{n,\delta})]ds = O(\delta)$,
as $\delta \to 0$ and $T \to \infty$.

Proof. The proof of this lemma is analogous to that of Lemma A.1 in Lu and Park (2018) by replacing B_T by $B_{n,\delta}$. □

Lemma A.2. *Let Γ_T be the sample autocovariance function of U, we have*
(a) $\sup_{s\in[-T,T]} |\Gamma_T(s)| = O(T_s^2)$,
(b) $\max_{|j|\leq n-1} \sup_{s\in[j\delta,(j+1)\delta]} |\Gamma_T(s) - \Gamma_T(j\delta)| = O(T_s \Delta_{\delta,T})$, *and*
(c) $\max_{|j|\leq n-1} |\Gamma_T(j\delta) - \gamma_n(j)| = O(T_s \Delta_{\delta,T})$,
as $\delta \to 0$ and $T \to \infty$, where $\Delta_{\delta,T} = \Delta_{\delta,T}(U)$ and $T_s = T(U)$.

Proof. Part (a) holds trivially as $\sup_{s\in[-T,T]} |\Gamma_T(s)| \leq T^{-1} \left(\sup_{0\leq t\leq T} U_t^2\right) \int_0^T dt = T(U^2) = O(T_s^2)$. Moreover, Part (b) holds because

$$\begin{aligned}
\max_{|j|\leq n-1} \sup_{s\in[j\delta,(j+1)\delta]} |\Gamma_T(s) - \Gamma_T(j\delta)| &\leq \frac{1}{T} \max_{|j|\leq n-1} \sup_{s\in[j\delta,(j+1)\delta]} \int_0^T |U_t (U_{t-s} - U_{t-j\delta})| \, dt \\
&\leq \frac{1}{T} \sup_{0\leq t\leq T} |U_t| \int_0^T \max_{|j|\leq n-1} \sup_{s\in[j\delta,(j+1)\delta]} |U_{t-s} - U_{t-j\delta}| dt \\
&\leq \sup_{0\leq t\leq T} |U_t| \left(\sup_{0\leq s,t\leq T} \sup_{|t-s|\leq\delta} |U_t - U_s| \right) = T_s \Delta_{\delta,T}.
\end{aligned}$$

As for Part (c), we first note that $\gamma_n(j) = T^{-1} \sum_{i=1}^n \int_{(i-1)\delta}^{i\delta} U_{i\delta} U_{(i-j)\delta} dt$, and then

$$\Gamma_T(j\delta) - \gamma_n(j) = A_{nj} + B_{nj},$$

where

$$A_{nj} = \frac{1}{T} \sum_{i=1}^n \int_{(i-1)\delta}^{i\delta} U_t (U_{t-j\delta} - U_{i\delta-j\delta}) \, dt \quad \text{and} \quad B_{nj} = \frac{1}{T} \sum_{i=1}^n \int_{(i-1)\delta}^{i\delta} (U_t - U_{i\delta}) U_{i\delta-j\delta} dt$$

for which we have

$$\max_{|j|\leq n-1}|A_{nj}| \leq \frac{1}{T}\sup_{0\leq t\leq T}|U_t| \max_{|j|\leq n-1}\sum_{i=1}^{n}\int_{(i-1)\delta}^{i\delta}\left(|U_{t-j\delta}-U_{i\delta-j\delta}|\right)dt$$
$$\leq \frac{1}{T}\sup_{0\leq t\leq T}|U_t|\left(T\sup_{0\leq t,s\leq T}\sup_{|t-s|<\delta}|U_t-U_s|\right) = T_s\Delta_{\delta,T},$$

and similarly, $\max_{|j|\leq n-1}|B_{nj}| \leq T_s\Delta_{\delta,T}$. This completes the proof of the lemma. □

LEMMA A.3. *Let Assumptions KF hold. For ω_n^2 and $\varpi_{n,\delta}^2$ defined in* (2.2) *and* (3.4) *respectively, we have*
(i) $\varpi_{n,\delta}^2 - \delta\omega_n^2 = O(\Delta_{\delta,T}B_{n,\delta})$, if U satisfies Assumption ST, and
(ii) $\varpi_{n,\delta}^2 - \delta\omega_n^2 = O(T_s\Delta_{\delta,T}B_{n,\delta} + T_s^2\delta)$, if U satisfies Assumption NS,
as $\delta \to 0$ and $T \to \infty$.

Proof. To show the lemma, we first write

$$\varpi_{n,\delta}^2 - \delta\omega_n^2 = P_{n,\delta} + Q_{n,\delta} + R_{n,\delta},$$

where

$$P_{n,\delta} = \delta\sum_{|j|\leq n-1}K(j\delta/B_{n,\delta})[\Gamma_T(j\delta) - \gamma_n(j)]$$
$$Q_{n,\delta} = \sum_{|j|\leq n-1}\int_{j\delta}^{(j+1)\delta}[K(s/B_{n,\delta})\Gamma_T(s) - K(j\delta/B_{n,\delta})\Gamma_T(j\delta)]ds$$
$$R_{n,\delta} = \int_{-n\delta}^{-(n-1)\delta}K(s/B_{n,\delta})\Gamma_T(s)ds,$$

which are the same as $P_{n,\delta}, Q_{n,\delta}$ and $R_{n,\delta}$ defined in the proof of Theorem 4.1 in Lu and Park (2018) except for B_T is replaced by $B_{n,\delta}$. The analysis of $P_{n,\delta}, Q_{n,\delta}$ and $R_{n,\delta}$ here is the same as that in the proof of Theorem 4.1 in Lu and Park (2018), while their stochastic orders are determined upon different assumptions for stationary U and nonstationary U.

If U satisfies Assumption ST, Lemma A.2 in Lu and Park (2018) holds under Assumption CA (i). By the proof of Theorem 4.1 in Lu and Park (2018), we have $P_{n,\delta} = O(\Delta_{\delta,T}B_{n,\delta})$, $Q_{n,\delta} = O(\Delta_{\delta,T}B_{n,\delta})$ and $R_{n,\delta} = O(\delta)$. So Part (i) of the lemma follows immediately. On the other hand, if U satisfies Assumption NS, Lemma A.2 is applicable. We may follow the analysis in the proof of Theorem 4.1 in Lu and Park (2018) and deduce that $P_{n,\delta} = O(B_{n,\delta}T_s\Delta_{\delta,T})$, $Q_{n,\delta} = O(B_{n,\delta}T_s\Delta_{\delta,T})$ and $R_{n,\delta} = O(T_s^2\delta)$, by Lemma A.1 and A.2.

This establishes Part (ii) of the lemma. $\square$

Lemma A.4. *For $\varpi_{n,\delta}^2$ defined in (3.4), we have*
(i) $B_{n,\delta}^{-1}\varpi_{n,\delta}^2 \to_p \sigma^2\iota(K)$, if U satisfies Assumption ST and $B_{n,\delta} \to_p 0$ with $\Delta_{\delta,T} \to_p 0$,
(ii) $c_T^{-2}B_{n,\delta}^{-1}\varpi_{n,\delta}^2 \to_p \iota(K)\int_0^1 U_t^{\circ 2}dt$, if U satisfies Assumption NS and $B_{n,\delta}/T \to_p 0$, and
(iii) $c_T^{-2}T^{-1}\varpi_{n,\delta}^2 \to_d \int_{-1}^1 K(s/B^\circ)\Gamma^\circ(s)ds$, if U satisfies Assumption NS and $B_{n,\delta}/T \to_d B^\circ \neq 0$ a.s.,
as $\delta \to 0$ and $T \to \infty$.

Proof. To show Part (i), we consider

$$B_{n,\delta}^{-1}\varpi_{n,\delta}^2 = B_{n,\delta}^{-1}\int_{-T}^{T} K(s/B_{n,\delta})\Gamma_T(s)ds = \int_{-T/B_{n,\delta}}^{T/B_{n,\delta}} K(s)\Gamma_T(sB_{n,\delta})ds.$$

Then we can choose $0 < \varepsilon < 1$ such that $T^\varepsilon B_{n,\delta} \prec_p \delta$ as $\delta \to 0$ and $T \to \infty$, and write

$$\int_{-T/B_{n,\delta}}^{T/B_{n,\delta}} K(s)\Gamma_T(sB_{n,\delta})ds - \sigma^2\iota(K) = P_{n,\delta} + Q_{n,\delta} + R_T$$

where

$$\begin{aligned} P_{n,\delta} &= \int_{|s|\le T^\varepsilon} K(s)\left[\Gamma_T(sB_{n,\delta}) - \sigma^2\right]ds, \\ Q_{n,\delta} &= \int_{T^\varepsilon \le |s| \le T/B_{n,\delta}} K(s)\Gamma_T(sB_{n,\delta})ds, \\ R_T &= -\sigma^2\int_{|s|\ge T^\varepsilon} K(s)ds \end{aligned}$$

each of which will be shown to be $o_p(1)$. For $P_{n,\delta}$, note that for $|s| \le T^\varepsilon$,

$$\begin{aligned} |\Gamma_T(sB_{n,\delta}) - \sigma^2| &= \left|\frac{1}{T}\int_0^T U_t(U_{t-sB_{n,\delta}} - U_t)dt + \left(\frac{1}{T}\int_0^T U_t^2dt - \sigma^2\right)\right| \\ &\le \frac{1}{T}\int_0^T |U_t(U_{t-sB_{n,\delta}} - U_t|dt + \left|\frac{1}{T}\int_0^T U_t^2dt - \sigma^2\right| \end{aligned} \tag{A.1}$$

where the second term on the right hand side of (A.1) is $o_p(1)$ by Assumption ST. As for the first term on the right hand side of (A.1), we note that for $|s| \leq T^{\varepsilon}$,

$$\begin{aligned}
\mathbb{E}\left|\frac{1}{T}\int_0^T U_t(U_{t-sB_{n,\delta}} - U_t)dt\right| &\leq \sup_{0\leq t\leq T} \mathbb{E}|U_t(U_{t-sB_{n,\delta}} - U_t)| \\
&\leq \left[\sup_{0\leq t\leq T} \mathbb{E}(U_t^2)\right]^{1/2} \left[\sup_{0\leq t\leq T} \mathbb{E}(U_{t-sB_{n,\delta}} - U_t)^2\right]^{1/2} \\
&\leq \left[\sup_{0\leq t\leq T} \mathbb{E}(U_t^2)\right]^{1/2} \left[\sup_{0\leq s,t\leq T}\ \sup_{|t-s|\leq T^{\varepsilon}B_{n,\delta}} \mathbb{E}(U_t - U_s)^2\right]^{1/2} \prec_p \Delta_{\delta,T}
\end{aligned}$$

by Assumption ST and the fact that $T^{\varepsilon}B_{n,\delta} \prec_p \delta$. Therefore, we have $1(|s| \leq T^{\varepsilon})[\Gamma_T(sB_{n,\delta}) - \sigma^2] \prec_p \Delta_{\delta,T} + o_p(1) \to_p 0$. Now, if we define a continuous functional $L(f) = \int_{-\infty}^{\infty} K(x)f(x)dx$ on the space of all continuous functions on $[-\infty, \infty]$ with supremum norm, then we may invoke the continuous mapping theorem (CMT) to deduce that

$$P_{n,\delta} = \int_{-\infty}^{\infty} K(s)1(|s| \leq T^{\varepsilon})[\Gamma_T(sB_{n,\delta}) - \sigma^2]ds \to_p 0$$

as $\delta \to 0$ and $T \to \infty$. As for $Q_{n,\delta}$, we have

$$\mathbb{E}|Q_{n,\delta}| \leq \int_{T^{\varepsilon}\leq|s|\leq T/B_{n,\delta}} |K(s)|\mathbb{E}|\Gamma_T(sB_{n,\delta})|ds \leq \sup_{s\in[-T,T]} \mathbb{E}|\Gamma_T(s)| \int_{|s|\geq T^{\varepsilon}} |K(s)|ds \to 0$$

as $T \to \infty$, since K is absolutely integrable and $\sup_{s\in[-T,T]} \mathbb{E}|\Gamma_T(s)| = O(1)$ as shown in Lemma A.2 in Lu and Park (2018). Lastly, $R_T \to 0$ as $T \to \infty$ simply because K is integrable.

The proof of Part (ii) and (iii) are essentially the same as that of Lemma 3.5 except for B_T is replaced by $B_{n,\delta}$, and hence is omitted here. □

LEMMA A.5. *We have*
(i) $\int_0^T \left(\int_0^t U_s ds\right)^2 dt - \delta^3 \sum_{i=1}^n \left(\int_{j=1}^i u_j\right)^2 \prec_p T^3(\Delta_{\delta,T} + \Delta_{\delta,T}^2)$, *if* U *satisfies Assumption ST, and*
(ii) $\int_0^T \left(\int_0^t U_s ds\right)^2 dt - \delta^3 \sum_{i=1}^n \left(\int_{j=1}^i u_j\right)^2 \prec_p T^2 T_s(\delta T_s + \Delta_{\delta,T} T)$, *if* U *satisfies Assumption NS,*
as $\delta \to 0$ *and* $T \to \infty$, *where* $\Delta_{\delta,T} = \Delta_{\delta,T}(U)$ *and* $T_s = T(U)$.

Proof. Note we have

$$\int_0^T \left(\int_0^t U_s ds\right)^2 dt - \delta^3 \sum_{i=1}^n \left(\sum_{j=1}^i u_j\right)^2 = R_T^a + R_T^b$$

where

$$R_T^a = -2\delta \sum_{i=1}^n \left[\left(\sum_{j=1}^i u_j\right)\left(\int_{(i-1)\delta}^{i\delta} A_{it} dt\right)\right] \quad \text{and} \quad R_T^b = \sum_{i=1}^n \int_{(i-1)\delta}^{i\delta} A_{it}^2 dt$$

with $A_{it} = \sum_{j=1}^i \int_{(j-1)\delta}^{j\delta} (U_{j\delta} - U_s) ds + \int_t^{i\delta} U_s ds$ for $i = 1, \cdots, n$ and $t \in [(i-1)\delta, i\delta]$.

First, we prove Part (i) of the lemma. If U satisfies Assumption ST, we analyze R_T^a as follows:

$$\begin{aligned}
\mathbb{E}\left|R_T^a\right| &\le 2\delta \sum_{i=1}^n \left[\mathbb{E}\left(\sum_{j=1}^i u_j\right)^2 \mathbb{E}\left(\int_{(i-1)\delta}^{i\delta} A_{it} dt\right)^2\right]^{1/2} \\
&\le 2\delta \sum_{i=1}^n \left[\mathbb{E}\left(\sum_{j=1}^i u_j\right)^2 \delta \int_{(i-1)\delta}^{i\delta} \mathbb{E}\left(A_{it}^2\right) dt\right]^{1/2} \\
&\le 2\delta^{3/2} \left[\max_{1\le i\le n} \mathbb{E}\left(\sum_{j=1}^i u_j\right)^2\right]^{1/2} \left[\max_{1\le i\le n} \sup_{t\in[(i-1)\delta, i\delta]} \mathbb{E}\left(A_{it}^2\right)\right]^{1/2} \sum_{i=1}^n \left[\int_{(i-1)\delta}^{i\delta} dt\right]^{1/2} \\
&= 2\delta T \left[\max_{1\le i\le n} \mathbb{E}\left(\sum_{j=1}^i u_j\right)^2\right]^{1/2} \left[\max_{1\le i\le n} \sup_{t\in[(i-1)\delta, i\delta]} \mathbb{E}\left(A_{it}^2\right)\right]^{1/2},
\end{aligned} \tag{A.2}$$

where the first and second inequalities follow from Cauchy Schwartz inequality of expectation and Cauchy Schwartz inequality of integration, respectively. Moreover, note that

$$\mathbb{E}\left|R_T^b\right| \le \left[\max_{1\le i\le n} \sup_{t\in[(i-1)\delta, i\delta]} \mathbb{E}\left(A_{it}^2\right)\right] \sum_{i=1}^n \int_{(i-1)\delta}^{i\delta} dt = T \max_{1\le i\le n} \sup_{t\in[(i-1)\delta, i\delta]} \mathbb{E}\left(A_{it}^2\right). \tag{A.3}$$

In particular, we have

$$\max_{1\le i\le n} \mathbb{E}\left(\sum_{j=1}^i u_j\right)^2 \le n^2 \sup_{0\le t\le T} \mathbb{E}(U_t^2) = O(\delta^{-2}T^2), \tag{A.4}$$

and for $i=1,\cdots,n$ and $t\in[(i-1)\delta,i\delta]$,

$$\mathbb{E}\left(A_{it}^2\right)\leq 2\mathbb{E}\left(\sum_{j=1}^{i}\int_{(j-1)\delta}^{j\delta}\left(U_{j\delta}-U_s\right)ds\right)^2+2\mathbb{E}\left(\int_t^{i\delta}U_s ds\right)^2 \tag{A.5}$$

where

$$\begin{aligned}\mathbb{E}\left(\sum_{j=1}^{i}\int_{(j-1)\delta}^{j\delta}\left(U_{j\delta}-U_s\right)ds\right)^2 &\leq i\mathbb{E}\left[\sum_{j=1}^{i}\left(\int_{(j-1)\delta}^{j\delta}\left(U_{j\delta}-U_s\right)ds\right)^2\right]\\ &\leq i\mathbb{E}\left[\sum_{j=1}^{i}\left(\int_{(j-1)\delta}^{j\delta}\left(U_{j\delta}-U_s\right)^2ds\right)\left(\int_{(j-1)\delta}^{j\delta}ds\right)\right]\\ &= i\delta\mathbb{E}\left[\sum_{j=1}^{i}\int_{(j-1)\delta}^{j\delta}\left(U_{j\delta}-U_s\right)^2ds\right]\end{aligned}$$

and $\mathbb{E}\left(\int_t^{i\delta}U_s ds\right)^2\leq\mathbb{E}\left(\int_t^{i\delta}U_s^2 ds\right)\left(\int_t^{i\delta}ds\right)\leq\delta\int_t^{i\delta}\mathbb{E}(U_s^2)ds$, from which it follows that

$$\begin{aligned}\max_{1\leq i\leq n}\sup_{t\in[(i-1)\delta,i\delta]}\mathbb{E}\left(A_{it}^2\right)\leq 2n\delta\mathbb{E}&\left[\sup_{0\leq s,t\leq T}\sup_{|t-s|\leq\delta}\left(U_t-U_s\right)^2\sum_{j=1}^{n}\int_{(j-1)\delta}^{j\delta}ds\right]\\ &+2\delta\sup_{0\leq t\leq T}\mathbb{E}(U_t)^2\int_{(i-1)\delta}^{i\delta}ds\sim_p\Delta_{\delta,T}^2T^2.\end{aligned} \tag{A.6}$$

Thus we can deduce $R_T^a\prec_p\Delta_{\delta,T}T^3$ from (A.2), (A.4) and (A.6), and $R_T^b\prec_p\Delta_{\delta,T}^2T^3$ from (A.3) and (A.6). Part (i) of the lemma then follows immediately.

Next, we prove Part (ii) of the lemma. When U satisfies Assumption NS, we consider

$$\begin{aligned}|R_T^a|&\leq 2\delta\sum_{i=1}^{n}\left|\sum_{j=1}^{i}u_j\right|\left|\int_{(i-1)\delta}^{i\delta}A_{it}dt\right|\leq 2\delta\left(n\sup_{0\leq t\leq T}|U_t|\right)\sum_{i=1}^{n}\left|\int_{(i-1)\delta}^{i\delta}A_{it}dt\right|\\ &\leq 2TT_s\sum_{i=1}^{n}\int_{(i-1)\delta}^{i\delta}|A_{it}|\,dt\leq 2T^2T_s\max_{1\leq i\leq n}\sup_{(i-1)\delta\leq t\leq i\delta}|A_{it}|\end{aligned} \tag{A.7}$$

and

$$\left|R_T^b\right|\leq\left(\max_{1\leq i\leq n}\sup_{(i-1)\delta\leq t\leq i\delta}A_{it}^2\right)\sum_{i=1}^{n}\int_{(i-1)\delta}^{i\delta}dt=T\max_{1\leq i\leq n}\sup_{(i-1)\delta\leq t\leq i\delta}A_{it}^2. \tag{A.8}$$

In this case, we simply have

$$
\begin{aligned}
\max_{1\le i\le n}\sup_{(i-1)\delta\le t\le i\delta}|A_{it}| &\le \sum_{j=1}^{n}\int_{(j-1)\delta}^{j\delta}|U_{j\delta}-U_s|ds+\max_{1\le i\le n}\int_{(i-1)\delta}^{i\delta}|U_s|ds\\
&\le T\sup_{0\le t,s\le T}\sup_{|t-s|<\delta}|U_t-U_s|+\delta\sup_{0\le t\le T}|U_t| = T\Delta_{\delta,T}+\delta T_s
\end{aligned}
$$

from which and (A.7) and (A.8) it follows that $R_T^a \prec_p T^3T_s\Delta_{\delta,T}+\delta T^2T_s^2$ and $R_T^b \prec_p T^3\Delta_{\delta,T}^2+\delta^2 TT_s^2$. So the result in Part (ii) of the lemma follows immediately from the fact that $T_s \succ_p \Delta_{\delta,T}$, and hence the whole lemma is proved. □

Lemma A.6. *Let Assumptions CA and NP hold. If $\theta_n(r)$ is the estimate of $\theta(r)$ under NP, and*

$$
\Theta_{n,\delta}(r) = \frac{\int_{|s|\le A_{n,\delta}}|s|^r\Gamma_T(s)ds}{\int_{|s|\le A_{n,\delta}}\Gamma_T(s)ds} \tag{A.9}
$$

where $A_{n,\delta}=\delta a_n$, then under Assumptions ST and NS, we have

$$
\delta^r\theta_n(r) \sim_p \Theta_{n,\delta}(r)
$$

as $\delta\to 0$ and $T\to\infty$.

Proof. If NP is used, then $\delta^r\theta_n(r)=\delta^{1+r}\sum_{|j|\le a_n}|j|^r\gamma_n(j)\Big/\delta\sum_{|j|\le a_n}\gamma_n(j)$. To prove the lemma, it suffices to show that for an integer $r\ge 0$,

$$
\int_{|s|\le A_{n,\delta}}|s|^r\Gamma_T(s)ds \sim_p \delta^{1+r}\sum_{|j|\le a_n}|j|^r\gamma_n(j) \tag{A.10}
$$

as $\delta\to 0$ and $T\to\infty$. To this end, we write

$$
\int_{|s|\le A_{n,\delta}}|s|^r\Gamma_T(s)ds-\delta^{1+r}\sum_{|j|\le a_n}|j|^r\gamma_n(j) = P_{n,\delta}+Q_{n,\delta}+R_{n,\delta} \tag{A.11}
$$

where $P_{n,\delta}, Q_{n,\delta}$ and $R_{n,\delta}$ are the same as those defined in the proof of Proposition 4.2 in Lu and Park (2018) with A_T replaced by $A_{n,\delta}$.

If U satisfies Assumption ST, Chang et al. (2018) has shown that (A.10) holds given $\Delta_{\delta,T}\to 0$ in their proof of Lemma 5.1. On the other hand, if U satisfies Assumption NS, then Lemma A.2 holds. We follow the analysis in the proof of Proposition 4.2 in Lu and Park (2018), and deduce that $P_{n,\delta}=O(T_s\Delta_{\delta,T}A_{n,\delta}^{1+r})$, $Q_{n,\delta}=O(T_s\Delta_{\delta,T}A_{n,\delta}^{1+r}+T_s^2\delta A_{n,\delta}^r)$ and

$R_{n,\delta} = O(T_s^2 \delta A_{n,\delta}^r)$ using Lemma A.2. Then it follows from (A.11) that

$$\int_{|s|\le A_{n,\delta}} |s|^r \Gamma_T(s)ds - \delta^{1+r} \sum_{|j|\le a_n} |j|^r \gamma_n(j) = O(T_s \Delta_{\delta,T} A_{n,\delta}^{1+r} + T_s^2 \delta A_{n,\delta}^r)$$

as $\delta \to 0$ and $T \to \infty$. Moreover, since $c_T^{-2} A_{n,\delta}^{-(1+r)} \int_{|s|\le A_{n,\delta}} |s|^r \Gamma_T(s)ds \to_d \int_{-1}^1 |s|^r (\int_0^1 U_t^{\circ 2} dt)ds$ as $\delta \to 0$ and $T \to \infty$, (A.10) holds by observing that $c_T^{-2} A_{n,\delta}^{-(1+r)} \big(\delta^{1+r} \sum_{|j|\le a_n} |j|^r \gamma_n(j) - \int_{|s|\le A_{n,\delta}} |s|^r \Gamma_T(s)ds\big) = O(c_T^{-2} T_s \Delta_{\delta,T} + c_T^{-2} T_s^2 \delta A_{n,\delta}^{-1}) = O\left(c_T^{-2} T_s (\Delta_{\delta,T} + T_s \delta^p T^{-p})\right) = o_p(1)$ under Assumption CA (ii) and Assumption NP. This completes the proof of the lemma. $\square$

Lemma A.7. *Let Assumptions CA and SP hold. If ρ_n is the estimated AR coefficient in the AR(1) regression of (u_i), then under both Assumptions ST and NS, we have*

$$1 - \rho_n \sim_p -\frac{\delta \int_0^T U_t dU_t}{\int_0^T U_t^2 dt}$$

as $\delta \to 0$ and $T \to \infty$.

Proof. Since $\rho_n - 1 = \sum_{i=1}^n u_{i-1}(u_i - u_{i-1}) / \sum_{i=1}^n u_{i-1}^2$, it suffices to show

$$\sum_{i=1}^n \delta u_i^2 \sim_p \int_0^T U_t^2 dt \tag{A.12}$$

and

$$\sum_{i=1}^n u_{i-1}(u_i - u_{i-1}) \sim_p \int_0^T U_t dU_t, \tag{A.13}$$

as $\delta \to 0$ and $T \to \infty$, under both Assumptions ST and NS.

To prove (A.12), note that under Assumption ST, we have $T^{-1} \int_0^T U_t^2 dt \to_p \sigma^2$ as $T \to \infty$, and

$$\begin{aligned}
\mathbb{E}\left|\frac{1}{T}\left(\int_0^T U_t^2 dt - \sum_{i=1}^n \delta u_{i-1}^2\right)\right| &\le \frac{1}{T} \sum_{i=1}^n \int_{(i-1)\delta}^{i\delta} \mathbb{E}\left|U_t^2 - U_{(i-1)\delta}^2\right| dt \\
&\le \frac{1}{T} \sum_{i=1}^n \int_{(i-1)\delta}^{i\delta} \left[\mathbb{E}\left(U_t + U_{(i-1)\delta}\right)^2\right]^{1/2} \left[\mathbb{E}\left(U_t - U_{(i-1)\delta}\right)^2\right]^{1/2} dt \\
&\le \left[2 \sup_{0\le t\le T} \mathbb{E}(U_t^2)\right]^{1/2} \left[\sup_{0\le s,t\le T} \sup_{|t-s|\le\delta} \mathbb{E}(U_t - U_s)^2\right]^{1/2} \frac{1}{T}\int_0^T dt \\
&= O(1)O(\Delta_{\delta,T}) = o(1)
\end{aligned}$$

under Assumptions CA (i). While under Assumption NS, we have $c_T^{-2}T^{-1}\int_0^T U_t^2 dt \to_d \int_0^1 U_t^{\circ 2} dt$ as $T \to \infty$, and

$$\begin{aligned}
\left|\int_0^T U_t^2 dt - \sum_{i=1}^n \delta u_{i-1}^2\right| &= \left|\sum_{i=1}^n \int_{(i-1)\delta}^{i\delta} (U_t^2 - U_{(i-1)\delta}^2) dt\right| \\
&\leq \sum_{i=1}^n \int_{(i-1)\delta}^{i\delta} \left|U_t + U_{(i-1)\delta}\right| |U_t - U_{(i-1)\delta}| dt \\
&\leq 2 \sup_{0\leq t\leq T} |U_t| \left(\sup_{0\leq s,t\leq T} \sup_{|t-s|\leq\delta} |U_t - U_s|\right) \sum_{i=1}^n \int_{(i-1)\delta}^{i\delta} dt = O(T_s \Delta_{\delta,T} T)
\end{aligned}$$

from which we deduce that $c_T^{-2}T^{-1}\left|\int_0^T U_t^2 dt - \sum_{i=1}^n \delta u_{i-1}^2\right| = O_p(c_T^{-2}T_s\Delta_{\delta,T}) = o_p(1)$ by Assumption CA (ii). This completes the proof of (A.12).

Next, to show (A.13), we consider

$$\begin{aligned}
\sum_{i=1}^n u_{i-1}(u_i - u_{i-1}) &= \frac{1}{2}(u_n^2 - u_0^2) - \frac{1}{2}\sum_{i=1}^n (u_i - u_{i-1})^2 \\
&= \frac{1}{2}\left[\left(U_T^2 - U_0^2\right) - \sum_{i=1}^n \left(U_{i\delta} - U_{(i-1)\delta}\right)^2\right] \qquad \text{(A.14)}
\end{aligned}$$

where by Itô formula, we have

$$\begin{aligned}
U_T^2 - U_0^2 &= 2\int_0^T U_t dU_t^c + [U^c]_T + \sum_{0\leq t\leq T} \Delta U_t^2 \\
&= 2\int_0^T U_t dU_t^c + [U^c]_T + \sum_{0\leq t\leq T} \left[2U_t \Delta U_t - (\Delta U_t)^2\right] \\
&= 2\left(\int_0^T U_t dU_t^c + \sum_{0\leq t\leq T} U_t \Delta U_t\right) + [U^c]_T - \sum_{0\leq t\leq T} (\Delta U_t)^2 \\
&= 2\int_0^T U_t dU_t + [U^c]_T - \sum_{0\leq t\leq T} (\Delta U_t)^2, \qquad \text{(A.15)}
\end{aligned}$$

where the second equality follows from $\Delta U_t^2 = U_t^2 - U_{t-}^2 = (U_t + U_{t-})(U_t - U_{t-}) = (2U_t -$

$\Delta U_t)\Delta U_t$. Moreover, we can write $(U_t - U_{(i-1)\delta})^2$ as

$$
\begin{aligned}
& 2\int_{(i-1)\delta}^{i\delta}(U_t - U_{(i-1)\delta})dU_t^c + \int_{(i-1)\delta}^{i\delta} d[U^c]_t + \sum_{(i-1)\delta\le t\le i\delta} \Delta\left(U_t - U_{(i-1)\delta}\right)^2 \\
=& 2\int_{(i-1)\delta}^{i\delta}(U_t - U_{(i-1)\delta})dU_t^c + \int_{(i-1)\delta}^{i\delta} d[U^c]_t + \sum_{(i-1)\delta\le t\le i\delta} \left[2\left(U_t - U_{(i-1)\delta}\right)\Delta U_t - (\Delta U_t)^2\right]
\end{aligned}
$$

by applying Itô formula again, and then deduce

$$
\sum_{i=1}^{n}\left(U_{i\delta} - U_{(i-1)\delta}\right)^2 = [U^c]_T - \sum_{0\le t\le T}(\Delta U_t)^2 + 2(R_T^a + R_T^b) \tag{A.16}
$$

where

$$
\begin{aligned}
R_T^a &= \sum_{i=1}^{n}\int_{(i-1)\delta}^{i\delta}(U_t - U_{(i-1)\delta})dU_t^c \\
R_T^b &= \sum_{i=1}^{n}\sum_{(i-1)\delta\le t\le i\delta}(U_t - U_{(i-1)\delta})\Delta U_t.
\end{aligned}
$$

Now from (A.14), (A.15) and (A.16) we obtain

$$
\int_0^T U_t dU_t - \sum_{i=1}^{n} u_i(u_i - u_{i-1}) = R_T^a + R_T^b \tag{A.17}
$$

which will be analyzed in a sequel.

To analyze R_T^a, we note that

$$
R_T^a = P_T + Q_T
$$

where

$$
\begin{aligned}
P_T &= \sum_{i=1}^{n}\int_{(i-1)\delta}^{i\delta}(U_t - U_{(i-1)\delta})dA_t \\
Q_T &= \sum_{i=1}^{n}\int_{(i-1)\delta}^{i\delta}(U_t - U_{(i-1)\delta})dM_t.
\end{aligned}
$$

under Assumption SP. For P_T, we have

$$|P_T| \leq \sup_{0\leq t,s\leq T}\sup_{|t-s|\leq\delta}|U_t - U_s|\sum_{i=1}^{n}\int_{(i-1)\delta}^{i\delta}|dA_t| = \Delta_{\delta,T}\int_0^T|dA_t| \preceq_p \Delta_{\delta,T}p_T T.$$

As for Q_T, it suffices to look at its quadratic variation

$$[Q]_T = \sum_{i=1}^{n}\int_{(i-1)\delta}^{i\delta}(U_t - U_{(i-1)\delta})^2 d[M]_t \leq \sup_{0\leq t,s\leq T}\sup_{|t-s|\leq\delta}(U_t - U_s)^2\int_0^T d[M]_t \preceq_p \Delta_{\delta,T}^2 q_T T$$

from which we deduce that $Q_T \preceq_p \Delta_{\delta,T}\sqrt{q_T T}$. Therefore, we have

$$R_T^a \preceq_p \Delta_{\delta,T}(p_T T + \sqrt{q_T T}).$$

Moreover, we have $R_T^b \preceq_p \Delta_{\delta,T}T$ because

$$\left|R_T^b\right| \leq \sup_{0\leq t,s\leq T}\sup_{|t-s|\leq\delta}|U_t - U_s|\sum_{i=1}^{n}\sum_{(i-1)\delta\leq t\leq i\delta}|\Delta U_t| = \Delta_{\delta,T}\sum_{0\leq t\leq T}|\Delta U_t| = \Delta_{\delta,T}O_p(T)$$

under Assumption SP. Now (A.17) implies that

$$T^{-1/2}\left[\int_0^T U_t dU_t - \sum_{i=1}^{n}u_i(u_i - u_{i-1})\right] \preceq_p \Delta_{\delta,T}(p_T\sqrt{T} + \sqrt{q_T}) = o_p(1)$$

under Assumption SP if U satisfies Assumption ST. On the other hand, if U satisfies Assumption NS, we have

$$c_T^{-2}\left[\int_0^T U_t dU_t - \sum_{i=1}^{n}u_i(u_i - u_{i-1})\right] \preceq_p c_T^{-2}\Delta_{\delta,T}(p_T T + \sqrt{q_T T}) = o_p(1).$$

This shows (A.13), and hence the proof of the lemma is complete. □

Lemma A.8. *Let Assumption CA and SP hold. If $\theta_n(r)$ is the estimate of $\theta(r)$ under SP, and $\Theta_T(r) = r!/\kappa_T^r$ under CSP, then under Assumptions ST and NS, we have*

$$\delta^r\theta_n(r) \sim_p \Theta_T(r)$$

as $\delta \to 0$ and $T \to \infty$.

Proof. Note $\theta_n(r)$ under SP is given as

$$\theta_n(r) = \frac{2(1-\rho_n)}{1+\rho_n} F_n(r), \tag{A.18}$$

where $F_n(r) = \sum_{j=1}^{\infty} j^r \rho_n^j$. By the use of Lemma A.7, we have

$$F_n(0) = \sum_{j=1}^{\infty} \rho_n^j = \frac{\rho_n}{1-\rho_n} \sim_p \frac{1-\delta\kappa_T}{\delta\kappa_T} = (\delta\kappa_T)^{-1} + o_p\big((\delta\kappa_T)^{-1}\big).$$

for $r = 0$, and

$$F_n(r) = \frac{1+(r-1)\rho_n}{1-\rho_n} F_n(r-1) + \frac{\rho_n}{1-\rho_n} \sum_{k=1}^{r-2} a_r(k) F_n(k) \tag{A.19}$$

$r \geq 1$, where the summation in (A.19) appears only when $r \geq 3$, and $a_r(k) = \sum_{i=1}^{k-1} \binom{r-2-i}{k-1-i}$ for $r \geq 3$. Note that

$$\begin{aligned} \frac{1+(r-1)\rho_n}{1-\rho_n} &\sim_p \frac{1+(r-1)(1-\delta\kappa_T)}{\delta\kappa_T} = \frac{r}{\delta\kappa_T} - (r-1) = r(\delta\kappa_T)^{-1} + o_p\big((\delta\kappa_T)^{-1}\big), \\ \frac{\rho_n}{1-\rho_n} &\sim_p \frac{1-\delta\kappa_T}{\delta\kappa_T} = \frac{1}{\delta\kappa_T} - 1 = (\delta\kappa_T)^{-1} + o_p\big((\delta\kappa_T)^{-1}\big). \end{aligned}$$

Therefore, the first term on the right hand side of (A.19) dominates, and we have for $r \geq 1$,

$$\begin{aligned} F_n(r) &= r(\delta\kappa_T)^{-1} F_n(r-1) + o_p\big((\delta\kappa_T)^{-1} F_n(r-1)\big) \\ &= r!(\delta\kappa_T)^{-r} F_n(0) + o_p\big((\delta\kappa_T)^{-r} F_n(0)\big) \\ &= r!(\delta\kappa_T)^{-(r+1)} + o_p\big((\delta\kappa_T)^{-(r+1)}\big). \end{aligned} \tag{A.20}$$

Now it follows from Lemma A.7, (A.18) and (A.20) that

$$\begin{aligned} \theta_n(r) &\sim_p \frac{2\delta\kappa_T}{2-\delta\kappa_T} \Big[r!(\delta\kappa_T)^{-(r+1)} + o_p\big((\delta\kappa_T)^{-(r+1)}\big) \Big] \\ &= [\delta\kappa_T + o_p(\delta\kappa_T)] \Big[r!(\delta\kappa_T)^{-(r+1)} + o_p\big((\delta\kappa_T)^{-(r+1)}\big) \Big] = r!(\delta\kappa_T)^{-r} + o_p\big((\delta\kappa_T)^{-r}\big) \end{aligned}$$

from which we deduce that

$$\delta^r \theta_n(r) = \frac{r!}{\kappa_T^r} + o_p(\kappa_T^{-r}) = \Theta_T(r)(1 + o_p(1))$$

as $\delta \to 0$ and $T \to \infty$, as desired. ∎

B. Proofs of Theorems

Proof of Lemma 3.1. To show the lemma, it suffices to show that

$$\delta\omega_n^2 \sim_p \varpi_{n,\delta}^2 \tag{B.1}$$

and

$$\delta^3 \sum_{i=1}^n \left(\sum_{j=1}^i u_j\right)^2 \sim_p \int_0^T \left(\int_0^t U_s ds\right)^2 dt \tag{B.2}$$

as $\delta \to 0$ and $T \to \infty$, under both Assumptions ST and NS.

To establish (B.1), we note that there are two cases under Assumption ST. The first is concerned with $B_{n,\delta} \to_p 0$ as $\delta \to 0$ sufficiently fast relative to $T \to \infty$, when RT or NP is used. In this case, we refer to Lemma A.4 (i) and deduce (B.1) by noticing that $B_{n,\delta}^{-1}\left(\delta\omega_n^2 - \varpi_{n,\delta}^2\right) = O(\Delta_{\delta,T}) = o_p(1)$ due to Lemma A.3 (i) and Assumption CA (i). Second, if $B_{n,\delta} \to_p \infty$ and $B_{n,\delta}/T \to_p 0$, as in the case when SP is used, we have $\varpi_{n,\delta}^2 \to_p \varpi^2$ as $\delta \to 0$ and $T \to \infty$. In this case, Lemma A.3 (i) and Assumption CA (i) imply that $\delta\omega_n^2 - \varpi_{n,\delta}^2 = O(\Delta_{\delta,T} B_{n,\delta}) = o(\Delta_{\delta,T} T) = o_p(1)$, as desired. There are also two cases to consider under Assumption NS, and the correponding asymptotics of $\varpi_{n,\delta}^2$ are given in Lemma A.4 (ii) and (iii). In both cases, we can prove (B.1) by the fact that $c_T^{-2} B_{n,\delta}^{-1}\left(\delta\omega_n^2 - \varpi_{n,\delta}^2\right) = O(c_T^{-2} T_s \Delta_{\delta,T} + c_T^{-2} B_{n,\delta}^{-1} T_s^2 \delta) = o_p(1)$ due to Lemma A.3 (ii) and Assumption CA (ii).

Next we show (B.2). If U satisfies Assumption ST, then $T^{-2}\int_0^T (\int_0^t U_s ds)^2 dt \to_d \int_0^1 U_t^{\circ 2} dt$ as $T \to \infty$. In this case, we have

$$T^{-2}\left[\delta^3 \sum_{i=1}^n \left(\sum_{j=1}^i u_j\right)^2 - \int_0^T \left(\int_0^t U_s ds\right)^2 dt\right] \prec_p T(\Delta_{\delta,T} + \Delta_{\delta,T}^2) = o_p(1)$$

by Lemma A.5 (i) and Assumption CA (i). Moreover, $c_T^{-2} T^{-3}\int_0^T (\int_0^t U_s ds)^2 dt \to_d \int_0^1 (\int_0^t U_s^{\circ 2} ds) dt$ as $T \to \infty$, if U satisfies Assumption NS. Then by Lemma A.5 (ii) and Assumption CA (ii), we have

$$c_T^{-2} T^{-3}\left[\delta^3 \sum_{i=1}^n \left(\sum_{j=1}^i u_j\right)^2 - \int_0^T \left(\int_0^t U_s ds\right)^2 dt\right] \prec_p c_T^{-2} T_s(\Delta_{\delta,T} + \delta T_s T^{-1}) = o_p(1),$$

which completes the proof of (B.2). $\square$

Proof of Lemma 3.2. If NP or SP is used, we have

$$B_{n,\delta} = \delta \left[\frac{r\pi^2(r)}{\iota(K^2)} \theta_n^2(r) \right]^{1/(2r+1)} n^{1/(2r+1)} = \left[\frac{r\pi^2(r)}{\iota(K^2)} \big(\delta^r \theta_n(r)\big)^2 \right]^{1/(2r+1)} T^{1/(2r+1)}$$

from which and Lemma A.6 and Lemma A.8 it follows that

$$B_{n,\delta} \sim_p \begin{cases} \left[\frac{r\pi^2(r)}{\iota(K^2)} \Theta_{n,\delta}^2(r) \right]^{1/(2r+1)} T^{1/(2r+1)}, \text{ if NP is used,} & \text{(B.3a)} \\ \left[\frac{r\pi^2(r)}{\iota(K^2)} \Theta_T^2(r) \right]^{1/(2r+1)} T^{1/(2r+1)}, \text{ if SP is used,} & \text{(B.3b)} \end{cases}$$

where $\Theta_{n,\delta}(r)$ is defined in (A.9) and $\Theta_T(r) = r!/\kappa_T^r$. Note that (B.3b) implies that when SP is used, $B_{n,\delta} \sim_p B_T$ where B_T is the continuous time bandwidth under CSP. Therefore, Part (b) of the lemma follows immediately from the definition of CSP and Lemma 3.4 (b).

For Part (a), the proof of the result for the stationary case is given in the proof of Lemma 5.1 in Chang et al. (2018), so here we only consider the case when U satisfies Assumption NS. Note that $\Theta_{n,\delta}(r)$ is the same as $\Theta_T(r)$ in (3.11) with A_T replaced by $A_{n,\delta} = \delta a_n$ such that $A_{n,\delta}/T = c\delta^{1-p}T^{p-1} \to 0$ as $\delta \to 0$ and $T \to \infty$. So, analogous to the proof of Lemma 3.4 (a), we have

$$A_{n,\delta}^{-r} \Theta_{n,\delta}(r) \to_d \frac{\int_{-1}^1 |s|^r (\int_0^1 U_t^{\circ 2} dt) ds}{\int_{-1}^1 (\int_0^1 U_t^{\circ 2} dt) ds} = \frac{1}{1+r},$$

from which and (B.3a), Part (a) of the lemma follows upon $A_{n,\delta} = c\delta^{1-p}T^p$. □

Proof of Theorem 3.3. By virtue of Lemma 3.1, it suffices to show the stated results for $\Lambda_{n,\delta}$. If RT or NP is used, $B_{n,\delta} \to_p 0$ as $\delta \to 0$ sufficiently fast relative to $T \to \infty$ under both Assumptions ST and NS. In this case, we invoke Lemma A.4 (i) to deduce that

$$B_{n,\delta}\Lambda_{n,\delta} = \frac{T^{-2} \int_0^T \left(\int_0^t U_s ds \right)^2 dt}{B_{n,\delta}^{-1} \varpi_{n,\delta}^2} = \frac{\int_0^1 \left(\frac{1}{\sqrt{T}} \int_0^{Tt} U_s ds \right)^2 dt}{B_{n,\delta}^{-1} \varpi_{n,\delta}^2} \to_d \frac{\varpi^2}{\sigma^2 \iota(K)} \int_0^1 W_t^2 dt,$$

as $\delta \to 0$ and $T \to \infty$ under Assumption ST. Moreover, Lemma A.4 (ii) implies that under Assumption NS,

$$B_{n,\delta}T^{-1}\Lambda_{n,\delta} = \frac{c_T^{-2} T^{-3} \int_0^T \left(\int_0^t U_s ds \right)^2 dt}{c_T^{-2} B_{n,\delta}^{-1} \varpi_{n,\delta}^2} = \frac{\int_0^1 \left(\int_0^t U_s^T ds \right)^2 dt}{c_T^{-2} B_{n,\delta}^{-1} \varpi_{n,\delta}^2} \to_d \frac{\int_0^1 \left(\int_0^t U_s^\circ ds \right)^2 dt}{\iota(K) \int_0^1 U_t^{\circ 2} dt},$$

as $\delta \to 0$ and $T \to \infty$. Therefore, we have $\Lambda_{n,\delta} \to_p \infty$ in both cases.

On the other hand, if SP is used, we know that $\varpi^2_{n,\delta} \to_p \varpi^2$ if U is stationary because SP scheme is high-frequency compatible. In this case, we have $\Lambda_{n,\delta} \to_d \int_0^1 W_t^2 dt$ under Assumption ST. If U is nonstationary satisfying Assumption NS, then Lemma 3.2 shows that $B_{n,\delta}/T \to_d B^\circ \neq 0$ a.s. In this case, we invoke Lemma A.4 (iii) to deduce that

$$\Lambda_{n,\delta} = \frac{c_T^{-2} T^{-3} \int_0^T \left(\int_0^t U_s ds \right)^2 dt}{c_T^{-2} T^{-1} \varpi^2_{n,\delta}} \to_d \frac{\int_0^1 \left(\int_0^t U_s^\circ ds \right)^2 dt}{\int_{-1}^1 K(s/B^\circ) \Gamma^\circ(s) ds}$$

as $\delta \to 0$ and $T \to \infty$. This completes the proof. $\square$

Proof of Lemma 3.4. If CNP is used, $B_T = c_T^*(r) T^{1/(1+2r)}$ and $\Theta_T(r)$ in $c_T^*(r)$ is defined in (3.11). By a change of variable, we have

$$\Theta_T(r) = \frac{\int_{-1}^1 |sA_T|^r \Gamma_T(sA_T) ds}{\int_{-1}^1 \Gamma_T(sA_T) ds} = \frac{A_T^r \int_{-1}^1 |s|^r \left(\int_0^1 c_T^{-1} U_{Tt} c_T^{-1} U_{T(t - sA_T/T)} dt \right) ds}{\int_{-1}^1 \left(\int_0^1 c_T^{-1} U_{Tt} c_T^{-1} U_{T(t - sA_T/T)} dt \right) ds},$$

which is followed by

$$A_T^{-r} \Theta_T(r) = \frac{\int_{-1}^1 |s|^r \left(\int_0^1 U_t^T U^T_{t - sA_T/T} dt \right) ds}{\int_{-1}^1 \left(\int_0^1 U_t^T U^T_{t - sA_T/T} dt \right) ds}, \tag{B.4}$$

where $U^T \to U^\circ$ as $T \to \infty$ under Assumption NS. Since $A_T = cT^p$ for $c > 0$ and $0 < p < 1$, we have $A_T/T \to 0$ as $T \to \infty$. Then for any $-1 \leq s \leq 1$, $U_t^T U^T_{t - sA_T/T} \to_d U_t^{\circ 2}$ on $[0, 1]$ as $T \to \infty$, and therefore $\int_0^1 U_t^T U^T_{t - sA_T/T} dt \to_d \int_0^1 U_t^{\circ 2} dt$. Now we define a continuous functional L as $L(f) = \int_{-1}^1 |x|^r f(x) dx$, $r \geq 0$, on the space of all continuous functions on $[-1, 1]$ with supremum norm and invoke CMT to deduce from (B.4) that

$$A_T^{-r} \Theta_T(r) \to_d \frac{\int_{-1}^1 |s|^r \left(\int_0^1 U_t^{\circ 2} dt \right) ds}{\int_{-1}^1 \left(\int_0^1 U_t^{\circ 2} dt \right) ds} = \frac{\int_{-1}^1 |s|^r ds}{\int_{-1}^1 ds} = \frac{1}{1+r}$$

as $T \to \infty$. Then Part (a) of the lemma follows immediately from the definition of B_T and $A_T = cT^p$.

If CSP is used, $B_T = c_T^*(r) T^{1/(1+2r)}$ and $\Theta_T(r) = r!/\kappa_T^r$ where $\kappa_T = -\int_0^T U_t dU_t / \int_0^T U_t^2 dt$.

Under Assumption NS, we have

$$c_T^{-2}\int_0^T U_t dU_t = \int_0^1 c_T^{-1}U_{Tt}dc_T^{-1}U_{Tt} = \int_0^1 U_t^T dU_t^T \to_d \int_0^1 U_t^\circ dU_t^\circ,$$
$$T^{-1}c_T^{-2}\int_0^T U_t^2 dt = \int_0^1 \left(c_T^{-1}U_{Tt}\right)^2 dt = \int_0^1 \left(U_t^T\right)^2 dt \to_d \int_0^1 U_t^{\circ 2}dt$$

as $T \to \infty$, from which it follows that $T\kappa_T \to_d -\int_0^1 U_t^\circ dU_t^\circ \big/ \int_0^1 U_t^{\circ 2}dt$ as $T \to \infty$, and this implies $T^{-r}\Theta_T(r) \to_d r!\left(\int_0^1 U_t^{\circ 2}dt \Big/ \int_0^1 U_t^\circ dU_t^\circ\right)^r$ as $T \to \infty$. Then Part (b) of the lemma follows directly. □

Proof of Lemma 3.5. To prove Part (a), note that by a change of variable, we have $\varpi_T^2 = B_T\int_{|s|\leq T/B_T} K(s)\Gamma_T(sB_T)ds$ where $\Gamma_T(sB_T) = \frac{1}{T}\int_0^T U_t U_{t-sB_T}dt = \int_0^1 U_{Tt}U_{T(t-sB_T/T)}dt$. Therefore, if we let $0 < \varepsilon < 1$ satisfy $B_T/T^{1-\varepsilon} \to_p 0$ as $T \to \infty$, we can write

$$c_T^{-2}B_T^{-1}\varpi_T^2 = Q_T + R_T \tag{B.5}$$

where

$$Q_T = \int_{|s|\leq T^\varepsilon} K(s)\left(\int_0^1 U_t^T U_{t-sB_T/T}^T dt\right)ds,$$
$$R_T = \int_{T^\varepsilon\leq|s|\leq T/B_T} K(s)\left(\int_0^1 U_t^T U_{t-sB_T/T}^T dt\right)ds.$$

We analyze Q_T first. Note that for $|s| \leq T^\varepsilon$, we have $sB_T/T \to_p 0$ jointly with $U^T \to_d U^\circ$ as $T \to \infty$, and therefore, $1(|s| \leq T^\varepsilon)U_t^T U_{t-sB_T/T}^T \to_d 1(-\infty \leq s \leq \infty)U_t^{\circ 2}$ as $T \to \infty$. Let L be a continuous functional defined as $L(f) = \int_0^1 f(x)dx$ on $D[0,1]$, then by CMT we have $1(|s| \leq T^\varepsilon)\int_0^1 U_t^T U_{t-sB_T/T}^T dt \to_d 1(-\infty \leq s \leq \infty)\int_0^1 U_t^{\circ 2}dt$ as $T \to \infty$. Moreover, given K is absolutely integrable, we may define another continuous functional L_K as $L_K(f) = \int_{-\infty}^\infty K(x)f(x)dx$ on a a space of continuous functions on $[-\infty,\infty]$ with the supremum norm, and then deduce from CMT again that

$$Q_T \to_d \int_{-\infty}^\infty K(s)\left(\int_0^1 U_t^{\circ 2}dt\right)ds = \iota(K)\int_0^1 U_t^{\circ 2}dt \tag{B.6}$$

as $T \to \infty$. As for R_T, we have

$$|R_T| \leq \sup_{0\leq t\leq 1}\left(U_t^T\right)^2\int_{|s|\geq T^\varepsilon}|K(s)|ds \to_p 0 \tag{B.7}$$

as $T \to \infty$, because K is absolutely integrable and $\sup_{0\le t\le 1}\left(U_t^T\right)^2 \to_d \sup_{0\le t\le 1} U_t^{\circ 2}$ as $T \to \infty$. Then Part (a) of the lemma holds due to (B.5)–(B.7).

Next, to show Part (b), we write $\varpi_T^2 = T\int_{-1}^1 K(sT/B_T)\Gamma_T(sT)ds$ where $\Gamma_T(sT) = \frac{1}{T}\int_0^T U_t U_{t-sT}dt = \int_0^1 U_{Tt}U_{T(t-s)}dt$. Then it follows that

$$c_T^{-2}T^{-1}\varpi_T^2 = \int_{-1}^1 K(sT/B_T)\left(\int_0^1 U_t^T U_{t-s}^T dt\right)ds.$$

If $B_T/T \to_d B^\circ \neq 0$ *a.s.* jointly with $U^T \to U^\circ$ as $T \to \infty$, then we have

$$c_T^{-2}T^{-1}\varpi_T^2 \to_d \int_{-1}^1 K(s/B^\circ)\left(\int_0^1 U_t^\circ U_{t-s}^\circ dt\right)ds = \int_{-1}^1 K(s/B^\circ)\Gamma^\circ(s)ds$$

as $T \to \infty$ by CMT. This completes the proof of Part (b) of the lemma. □

Proof of Proposition 3.6. Under Assumption NS, we have

$$c_T^{-2}T^{-3}\int_0^T\left(\int_0^t U_s ds\right)^2 dt = \int_0^1\left(\int_0^t U_s^T ds\right)^2 dt \to_d \int_0^1\left(\int_0^t U_s^\circ ds\right)^2 dt$$

as $T \to \infty$, from which and Lemma 3.5, Proposition 3.6 follow immediately. □

Proof of Lemma 4.1. First, note that by the same proof of Lemma 3.1 with $B_{n,\delta}$ replaced by B_T, we can show that under Assumption KF and CA, the KPSS stationarity test statistic λ_n in (2.1) with a high-frequency compatible bandwidth is asymptotically equivalent to Λ_T defined in (3.8). Second, the RB-KPSS test statistic is the same as λ_n with (u_i) is replaced by $(v_i - \bar{v}_n)$, and the corresponding continuous time RB-KPSS test statistic is the same as Λ_T with U replaced by $V - \bar{V}$. Therefore, to establish Lemma 4.1, it suffices to show that the $V - \bar{V}$ satisfies Assumption CA for both stationarity and cointegration tests as follows.

For the stationarity test, we have $v_i = u_i$ and $V = U$. In this case, the process $V - \bar{V} = U - \bar{U}$ satisfies Assumption CA trivially if U itself saftisfies Assumption CA. In specific, we have $\Delta_{\delta,T}(U - \bar{U}) = \Delta_{\delta,T}(U)$ and $T(U - \bar{U}) \sim_p T(U)$. Also, given $\sup_{0\le t\le\infty}\mathbb{E}(U_t^2) < \infty$, we have $\sup_{0\le t\le\infty}\mathbb{E}(U_t - \bar{U})^2 < \infty$. As for the cointegration test, $V_t = U_t - N_T X_t$ where

$$N_T = \frac{\int_0^T (X_t - \bar{X}_T)U_t dt}{\int_0^T (X_t - \bar{X}_T)^2 dt}. \tag{B.8}$$

If Assumption CI holds, we have $N_T \sim_p T^{-1/2}c_T^{-1}$, and then

$$\begin{aligned}\sup_{0\leq t\leq\infty} \mathbb{E}V_t^2 &\leq 2 \sup_{0\leq t\leq\infty} \mathbb{E}U_t^2 + 2 \lim_{T\to\infty} \sup_{0\leq t\leq T} \mathbb{E}(N_T X_t)^2 \\ &= 2 \sup_{0\leq t\leq\infty} \mathbb{E}U_t^2 + 2\mathbb{E} \lim_{T\to\infty} \sup_{0\leq t\leq T} (N_T X_t)^2 = O(1) + O(T^{-1}) = O(1)\end{aligned}$$

under Assumption CA′ (i). Moreover, notice that

$$\begin{aligned}\Delta_{\delta,T}(V) &= \sup_{0\leq s,t\leq T} \sup_{|t-s|\leq\delta} |V_t - V_s| \leq \sup_{0\leq s,t\leq T} \sup_{|t-s|\leq\delta} |U_t - U_s| + N_T \sup_{0\leq s,t\leq T} \sup_{|t-s|\leq\delta} |X_t - X_s| \\ &= \Delta_{\delta,T}(U) + N_T\Delta_{\delta,T}(X) \sim_p \Delta_{\delta,T}(U) + T^{-1/2}c_T^{-1}\Delta_{\delta,T}(X).\end{aligned}$$

Therefore, Assumption CA′ (i) guarantees V to satisfy Assumption CA (i). Next, if Assumption NC holds, we have $N_T \sim_p c_T d_T^{-1}$. For this case, we have $\Delta_{\delta,T}(V) \leq \Delta_{\delta,T}(U) + N_T\Delta_{\delta,T}(X) \sim_p \Delta_{\delta,T}(U) + c_T d_T^{-1}\Delta_{\delta,T}(X)$ and $T(V) = \sup_{0\leq t\leq T} |V_t| \leq \sup_{0\leq t\leq T} |U_t| + N_T \sup_{0\leq t\leq T} |X_t| \sim_p T(U) + c_T d_T^{-1} T(X)$ from which we can see that, again, the conditions in Assumption CA′ (ii) guarantee V to satisfy Assumption CA (ii). This completes the proof of the lemma. □

Proof of Theorem 4.2. By virtue of Lemma 4.1, it suffices to show that

$$\Lambda_T = \frac{\int_0^T \left(\int_0^t (V_s - \bar{V}_T)ds\right)^2 dt}{T^2 \int_{|s|\leq T} K(s/B_T)\Gamma_T(s)ds} \to_p \int_0^1 \left(\int_0^t dW_s^\circ\right)^2 dt \tag{B.9}$$

as $T\to\infty$. For stationarity test, we have $V_t = U_t$, and therefore,

$$\frac{1}{T^2}\int_0^T \left(\int_0^t (V_s - \bar{V}_T)ds\right)^2 dt = \int_0^1 \left(\frac{1}{\sqrt{T}}\int_0^{tT} \left(U_s - \bar{U}_T\right) ds\right)^2 dt \tag{B.10}$$

where under Assumption ST we have

$$\frac{1}{\sqrt{T}}\int_0^{tT} (U_s - \bar{U}_T)ds = \frac{1}{\sqrt{T}}\int_0^{tT} U_s ds - t\cdot\frac{1}{\sqrt{T}}\int_0^T U_s ds \to_d U_t^\circ - tU_1^\circ,$$

as $T\to\infty$, where U° is Brownian motion with longrun variance ϖ^2. Moreover, Lu and Park (2018) has shown that if $B_T\to\infty$ and $B_T/T\to 0$ as $T\to\infty$, we have $\int_{|s|\leq T} K(s/B_T)\Gamma_T(s)ds \to_p \varpi^2$ as $T\to\infty$. Hence (B.9) follows immediately in this case.

For the cointegration test, we have $V_t = U_t - N_T X_t$ where N_T is defined in (B.8). Under the Assumption CI, it follows that

$$c_T\sqrt{T}N_T = \frac{c_T^{-1}T^{-1/2}\int_0^T (X_t-\bar{X}_T)U_t dt}{c_T^{-2}T^{-1}\int_0^T (X_t-\bar{X}_T)^2 dt} = \frac{\int_0^1 (X_t^T-\bar{X}_1^T)dU_t^T}{\int_0^1 (X_t^T-\bar{X}_1^T)^2 dt} \to_d N^\circ \tag{B.11}$$

as $T\to\infty$, where $N^\circ = \varpi\int_0^1 (X_t^\circ-\bar{X}_1^\circ)dW_t/\int_0^1 (X_t^\circ-\bar{X}_1^\circ)^2 dt$. We first analyze the numerator of Λ_T as follows.

$$\frac{1}{T^2}\int_0^T \left(\int_0^t (V_s-\bar{V}_T)ds\right)^2 dt = \int_0^1 \left(\frac{1}{\sqrt{T}}\int_0^{tT} (V_s-\bar{V}_T)ds\right)^2 dt = \int_0^1 (Y_t+Z_t)^2 dt$$

where $Y_t = \frac{1}{\sqrt{T}}\int_0^{tT}(U_s-\bar{U}_T)ds \to_d U_t^\circ - tU_1^\circ = \varpi W_t - tW_1$, and

$$\begin{aligned} Z_t &= \frac{1}{\sqrt{T}}\int_0^{tT} N_T(X_s-\bar{X}_T)ds = c_T\sqrt{T}N_T\frac{1}{c_T T}\int_0^{tT}(X_s-\bar{X}_T)ds \\ &= c_T\sqrt{T}N_T\int_0^t (X_s^T-\bar{X}_1^T)ds \to_d N^\circ\int_0^t (X_s^\circ-\bar{X}_1^\circ)ds \end{aligned}$$

as $T\to\infty$ by Assumption CI. Hence the theorem holds if the denominator of Λ_T satisfies $\int_{|s|\le T} K(s/B_T)\Gamma_T(s)ds \to_p \varpi^2$ as $T\to\infty$ under Assumption CI. To see this, we write

$$\Gamma_T(s) = \frac{1}{T}\int_0^T (V_t-\bar{V}_T)(V_{t-s}-\bar{V}_T)dt = \tilde{\Gamma}_T(s) + G_T(s),$$

where $\tilde{\Gamma}_T(s) = T^{-1}\int_0^T (U_t-\bar{U}_T)(U_{t-s}-\bar{U}_T)dt$ and

$$G_T(s) = T^{-1}\left(P_T(s)+Q_T(s)+R_T(s)\right)$$

where

$$\begin{aligned} P_T(s) &= N_T^2\int_0^T (X_t-\bar{X}_T)(X_{t-s}-\bar{X}_T)dt \\ Q_T(s) &= N_T\int_0^T (X_t-\bar{X}_T)(U_{t-s}-\bar{U}_T)dt \\ R_T(s) &= N_T\int_0^T (U_t-\bar{U}_T)(X_{t-s}-\bar{X}_T)dt. \end{aligned}$$

As U satisfies Assumption ST under Assumption CI, we have $\int_{|s|\le T} K(s/B_T)\tilde{\Gamma}_T(s)ds \to_p \varpi^2$

as $T \to \infty$. Moreover, we have

$$\frac{T}{B_T}\int_{|s|\le T} K(s/B_T)G_T(s)ds = \int_{|s|\le T/B_T} K(s)\left[P_T(sB_T)+Q_T(sB_T)+R_T(sB_T)\right]ds$$

where

$$\begin{aligned}
\int_{|s|\le T/B_T} K(s)P_T(sB_T)ds &= \left(c_T\sqrt{T}N_T\right)^2 \int_{|s|\le T/B_T} K(s)\left[\int_0^1 (X_t^T-\bar{X}_1^T)(X_{t-sB_T/T}^T-\bar{X}_1^T)dt\right]ds \\
&\to_d N^{\circ 2}\iota(K)\int_0^1 (X_t^\circ-\bar{X}_1^\circ)^2 dt \\
\int_{|s|\le T/B_T} K(s)Q_T(sB_T)ds &= c_T\sqrt{T}N_T \int_{|s|\le T/B_T} K(s)\left[\int_0^1 (X_t^T-\bar{X}_1^T)(dU_{t-sB_T/T}^T-U_1^T dt)\right]ds \\
&\to_d N^{\circ 2}\iota(K)\int_0^1 (X_t^\circ-\bar{X}_1^\circ)(dU_t^\circ-U_1^\circ dt) \\
\int_{|s|\le T/B_T} K(s)R_T(sB_T)ds &= c_T\sqrt{T}N_T \int_{|s|\le T/B_T} K(s)\left[\int_0^1 (X_{t-sB_T/T}^T-\bar{X}_1^T)(dU_t^T-U_1^T dt)\right]ds \\
&\to_d N^{\circ 2}\iota(K)\int_0^1 (X_t^\circ-\bar{X}_1^\circ)(dU_t^\circ-U_1^\circ dt)
\end{aligned}$$

as $T \to \infty$, from which we deduce that $\int_{|s|\le T} K(s/B_T)G_T(s)ds = O_p(B_T/T) = o_p(1)$ as desired. This completes the proof of the thorem under Assumption CI. □

Proof of Lemma 4.3. For the RB-KPSS test, schemes CNP and CSP are the same as defined in Section 3.1 except for the process U is replaced by $V-\bar{V}_T$, where V is defined separately for stationarity and cointegration test. Since we have $c_T^{-1}(V_{Tt}-\bar{V}_T) \to_d V_t^\circ - \bar{V}_1$ as $T \to \infty$ on $D[0,1]$ under Assumption NS or NC, Lemma 4.3 follows immediately from Lemma 3.4 with U replaced by $V-\bar{V}_T$ and U° replaced by $V^\circ-\bar{V}_1^\circ$. □

Proof of Theorem 4.4. Note that under Assumption NS for stationarity test or under Assumption NC for cointegration test, we have $c_T^{-1}(V_{Tt}-\bar{V}_T) \to_d V_t^\circ - \bar{V}_1$ as $T \to \infty$ on $D[0,1]$. Therefore under Assumption NS or NC, Proposition 3.6 holds for continuous time RB-KPSS test statistic Λ_T with U replaced by $V-\bar{V}_T$ and U° replaced by $V^\circ-\bar{V}_1^\circ$, and then Theorem 4.4 can be deduced from Lemma 4.1. □

Proof of Corollary 4.5. Under Assumption CA for stationarity test or Assumption CA′ for cointegration test, the modified test statistic

$$\lambda_n^* \sim_p \frac{1}{T^2}\int_0^T \left(\int_0^t (V_s-\bar{V}_T)ds\right)^2 dt$$

as $\delta \to 0$ and $T \to \infty$, due to Lemma 4.1. Therefore, Part (a) of the corollary follows from the analysis of the numerator of Λ_T in the proof of Theorem 4.2. Moreover, Part (b) of the corollary holds because

$$\frac{1}{c_T^2 T^3} \int_0^T \left(\int_0^t (V_s - \bar{V}_T) ds \right)^2 dt = \int_0^1 \left(\int_0^t c_T^{-1} (V_{Ts} - \bar{V}_T) ds \right)^2 dt \to_d \int_0^1 \left(\int_0^t (V_s^\circ - \bar{V}_1^\circ) ds \right)^2 dt$$

as $T \to \infty$. $\square$

Proof of Corollary 6.1. Note that we have shown in (B.1) that $\delta \omega_n^2 \sim_p \varpi_{n,\delta}^2$. Therefore, to show $\sqrt{T} \lambda_n^{\mathrm{RS}} \sim_p \Lambda_{n,\delta}^{\mathrm{RS}}$, it suffices to prove that

$$\delta \left[\max_{1 \le i \le n} \sum_{j=1}^i u_j - \min_{1 \le i \le n} \sum_{j=1}^i u_j \right] \sim_p \max_{0 \le t \le 1} \int_0^{tT} U_s ds - \min_{0 \le t \le 1} \int_0^{tT} U_s ds. \tag{B.12}$$

To show this, note that for any $i = 1, \ldots, n$, and $t \in ((i-1)\delta/T, i\delta/T]$,

$$\int_0^{tT} U_s ds - \delta \sum_{j=1}^i u_j = \sum_{j=1}^i \int_{(j-1)\delta}^{j\delta} (U_s - U_{j\delta}) ds - \int_{tT}^{i\delta} U_s ds$$

from which we deduce that

$$\begin{aligned} \left| \max_{0 \le t \le 1} \int_0^{tT} U_s ds - \max_{1 \le i \le n} \delta \sum_{j=1}^i u_j \right| &\le \max_{1 \le i \le n} \max_{\frac{(i-1)\delta}{T} \le t \le \frac{i\delta}{T}} \left| \sum_{j=1}^i \int_{(j-1)\delta}^{j\delta} (U_s - U_{j\delta}) ds - \int_{tT}^{i\delta} U_s ds \right| \\ &\le \max_{1 \le i \le n} \left| \sum_{j=1}^i \int_{(j-1)\delta}^{j\delta} (U_s - U_{j\delta}) ds \right| + \max_{1 \le i \le n} \max_{\frac{(i-1)\delta}{T} \le t \le \frac{i\delta}{T}} \left| \int_{tT}^{i\delta} U_s ds \right| \\ &\le \Delta_{\delta,T} T + T_s \delta. \end{aligned}$$

Since the first inequality also applies to the case if we replace the maximal signs on the left hand of the inequality by the minimal signs, we can readily deduce that

$$\begin{aligned} &\left| \left(\max_{0 \le t \le 1} \int_0^{tT} U_s ds - \min_{0 \le t \le 1} \int_0^{tT} U_s ds \right) - \delta \left(\max_{1 \le i \le n} \sum_{j=1}^i u_j - \min_{1 \le i \le n} \sum_{j=1}^i u_j \right) \right| \\ \le &\left| \max_{0 \le t \le 1} \int_0^{tT} U_s ds - \max_{1 \le i \le n} \delta \sum_{j=1}^i u_j \right| + \left| \min_{0 \le t \le 1} \int_0^{tT} U_s ds - \min_{1 \le i \le n} \delta \sum_{j=1}^i u_j \right| = O_p(\Delta_{\delta,T} T + T_s \delta). \end{aligned}$$

In the case when U is stationary and satisfying Assumption ST, we have

$$\frac{1}{\sqrt{T}}\left(\max_{0\leq t\leq 1}\int_0^{tT}U_s ds - \min_{0\leq t\leq 1}\int_0^{tT}U_s ds\right)\to_d \max_{0\leq t\leq 1}U_t^\circ - \min_{0\leq t\leq 1}U_t^\circ. \tag{B.13}$$

Therefore, (B.12) holds if $\Delta_{\delta,T}T^{1/2}+T_sT^{-1/2}\delta\to_p 0$ as $\delta\to 0$ and $T\to\infty$ which is satisfied under the assumption that $T_sT^{-1}\delta\prec_p\Delta_{\delta,T}$ and Assumption CA (i). On the other hand, if U is nonstationary satisfying Assumption NS, it follows that

$$\frac{1}{Tc_T}\left(\max_{0\leq t\leq 1}\int_0^{tT}U_s ds - \min_{0\leq t\leq 1}\int_0^{tT}U_s ds\right)\to_d \max_{0\leq t\leq 1}\int_0^t U_s^\circ ds - \min_{0\leq t\leq 1}\int_0^t U_s^\circ ds, \tag{B.14}$$

and thus (B.12) holds given $\Delta_{\delta,T}c_T^{-1}+c_T^{-1}T_sT^{-1}\delta\to_p 0$ is satisfied by Assumption CA (ii).

Next, we show $\lambda_n^{\mathrm{VS}}\sim_p\Lambda_{n,\delta}^{\mathrm{VS}}$. Given (B.1) and Lemma 3.1, it is sufficient to prove that

$$\delta^2\sum_{i=1}^n\sum_{j=1}^i u_j\sim_p\int_0^T\int_0^t U_s dsdt, \tag{B.15}$$

for which we consider

$$\int_0^T\int_0^t U_s dsdt-\delta^2\sum_{i=1}^n\sum_{j=1}^i u_j=\delta\sum_{i=1}^n\sum_{j=1}^i\int_{(j-1)\delta}^{j\delta}(U_{j\delta}-U_s)ds-\sum_{i=1}^n\int_{(i-1)\delta}^{i\delta}\int_t^{i\delta}U_s dsdt.$$

Note that $\left|\delta\sum_{i=1}^n\sum_{j=1}^i\int_{(j-1)\delta}^{j\delta}(U_{j\delta}-U_s)ds\right|\leq\Delta_{\delta,T}\delta\sum_{i=1}^n\sum_{j=1}^i\int_{(j-1)\delta}^{j\delta}ds\leq\Delta_{\delta,T}\delta^2n^2=\Delta_{\delta,T}T^2$, and

$$\mathbb{E}\left|\sum_{i=1}^n\int_{(i-1)\delta}^{i\delta}\int_t^{i\delta}U_s dsdt\right|\leq\sup_{0\leq t\leq\infty}\mathbb{E}|U_t|\sum_{i=1}^n\int_{(i-1)\delta}^{i\delta}\int_t^{i\delta}dsdt=O(\delta T)$$

for the case when U satisfies Assumption ST, and

$$\left|\sum_{i=1}^n\int_{(i-1)\delta}^{i\delta}\int_t^{i\delta}U_s dsdt\right|\leq T_s\sum_{i=1}^n\int_{(i-1)\delta}^{i\delta}\int_t^{i\delta}dsdt=\delta TT_s$$

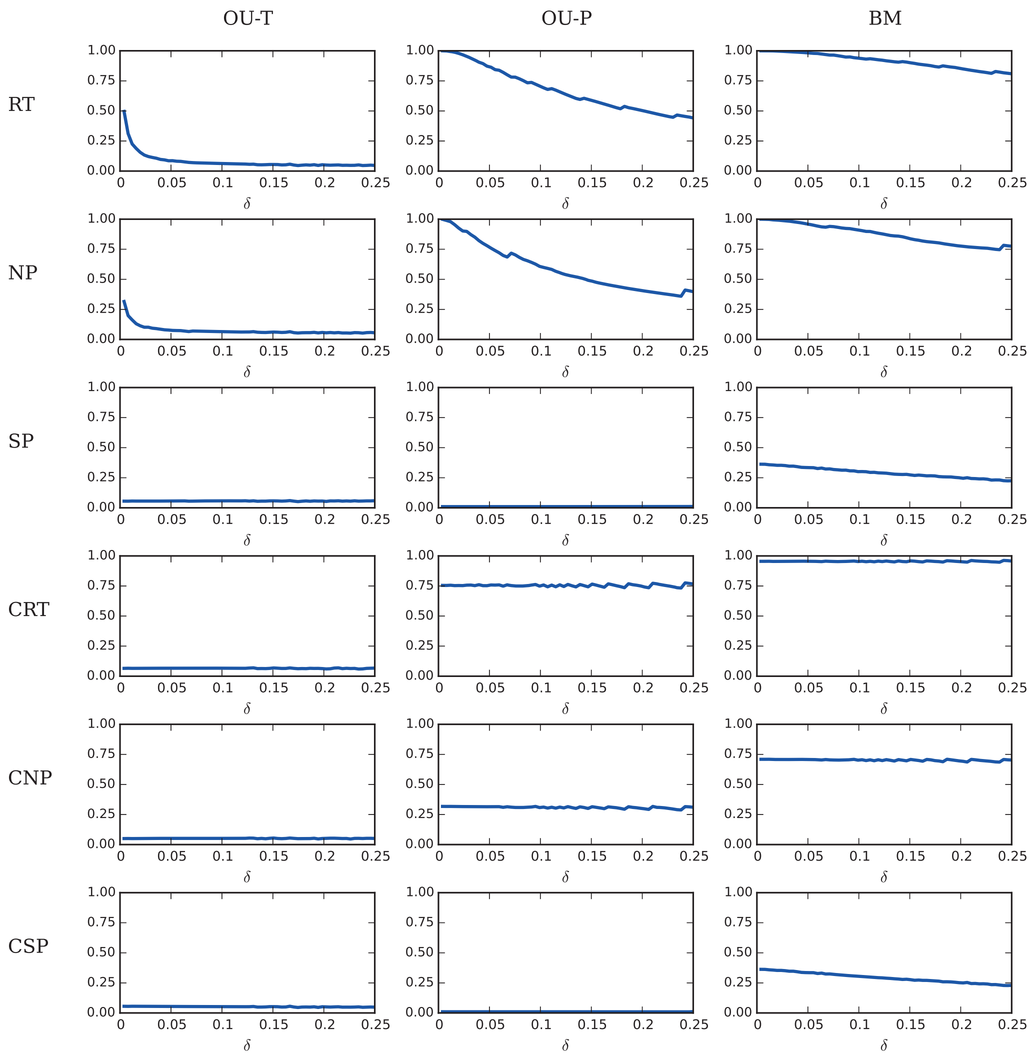
for the case when U satisfies Assumption NS. Therefore, we can deduce that $\int_0^T\int_0^t U_s dsdt-\delta^2\sum_{i=1}^n\sum_{j=1}^i u_j$ is $O_p(\Delta_{\delta,T}T^2+\delta T)$ and $O_p(\Delta_{\delta,T}T^2+\delta TT_s)$ under Assumption ST and NS, respectively. Since $T^{-3/2}\int_0^T\int_0^t U_s dsdt\to_d\int_0^1 U_t^\circ dt$ under Assumption ST, (B.15) holds because $T^{-3/2}\left[\int_0^T\int_0^t U_s dsdt-\delta^2\sum_{i=1}^n\sum_{j=1}^i u_j\right]=O_p(\Delta_{\delta,T}T^{1/2})=o_p(1)$ under Assumption CA (i). On the other hand, $c_T^{-1}T^{-2}\int_0^T\int_0^t U_s dsdt\to_d\int_0^1\int_0^t U_s^\circ dsdt$ under Assumption NS, so (B.15) holds in this case as $c_T^{-1}T^{-2}\left[\int_0^T\int_0^t U_s ds dt - \delta^2\sum_{i=1}^n\sum_{j=1}^i u_j\right] = O_p(c_T^{-1}\Delta_{\delta,T} + \delta c_T^{-1}T^{-1}T_s) = o_p(1)$ under Assumption CA (ii). This completes the proof of the corollary. $\square$

Proof of Corollary 6.2. We prove this corollary in line with the proof of Theorem 3.3. First, if RT or NP is used, then $B_{n,\delta}\to_p 0$ under both Assumption ST and NS. If U satisfies Assumption ST, then we have $\sqrt{B_{n,\delta}/T}\Lambda_{n,\delta}^{\mathrm{RS}}\to_d \varpi(M^\circ - m^\circ)/\sigma\sqrt{\iota(K)}$ where $M^\circ = \max_{0\le t\le 1} W_t$ and $m^\circ = \min_{0\le t\le 1} W_t$, by (B.13) and Lemma A.4 (i). This implies $\lambda_n^{\mathrm{RS}}\sim_p T^{-1/2}\Lambda_{n,\delta}^{\mathrm{RS}} = O_p(B_{n,\delta}^{-1/2})\to_p\infty$. As for λ_n^{VS}, we have $B_{n,\delta}\Lambda_{n,\delta}^{\mathrm{VS}}\to_d\left[\int_0^1 U_t^{\circ 2}dt - (\int_0^1 U_t^\circ dt)^2\right]/\sigma^2\iota(K)$, which implies $\lambda_n^{\mathrm{VS}}\sim_p\Lambda_{n,\delta}^{\mathrm{VS}} = O_p(B_{n,\delta}^{-1})\to_p\infty$. If U satisfies Assumption NS, Lemma A.4 (ii) and (B.14) yield that $\sqrt{B_{n,\delta}}/T\Lambda_{n,\delta}^{\mathrm{RS}}\to_d(\tilde{M}^\circ - \tilde{m}^\circ)/\sqrt{\iota(K)\int_0^1 U_t^{\circ 2}dt}$ where $\tilde{M}^\circ = \max_{0\le t\le 1}\int_0^t U_s^\circ ds$ and $\tilde{m}^\circ = \min_{0\le t\le 1}\int_0^t U_s^\circ ds$. This implies $\lambda_n^{\mathrm{RS}}\sim_p T^{-1/2}\Lambda_{n,\delta}^{\mathrm{RS}} = O_p(\sqrt{T/B_{n,\delta}})\to_p\infty$. For λ_n^{VS}, we have $B_{n,\delta}T^{-1}\Lambda_{n,\delta}^{\mathrm{VS}}\to_d\left[\int_0^1(\int_0^t U_s^\circ ds)^2 dt - (\int_0^1\int_0^t U_s^\circ ds dt)^2\right]/\iota(K)\int_0^1 U_t^{\circ 2}dt$, and hence $\lambda_n^{\mathrm{VS}}\sim_p\Lambda_{n,\delta}^{\mathrm{VS}} = O_p(T/B_{n,\delta})\to_p\infty$, as desired.

On the other hand, if SP is used, we have $\varpi_{n,\delta}\to_d\varpi$ if U is stationary. Then by Corollary 6.1 and (B.13) we have $\lambda_n^{\mathrm{RS}}\sim_p T^{-1/2}\Lambda_{n,\delta}^{\mathrm{RS}}\to_d M^\circ - m^\circ$, and for λ_n^{VS} we have $\lambda_n^{\mathrm{VS}}\sim_p\Lambda_{n,\delta}^{\mathrm{VS}}\to_d\int_0^1 W_t^2 dt - \left(\int_0^1 W_t dt\right)^2$ under Assumption ST. While if U is nonstationary satisfying Assumption NS, we have $c_T^{-2}T^{-1}\varpi_{n,\delta}^2\to_d\int_{-1}^1 K(s/B^\circ)\Gamma^\circ(s)ds$ as in Lemma 3.5 (b). Then it follows from Corollary 6.1 and (B.14) that $\lambda_n^{\mathrm{RS}}\sim_p T^{-1/2}\Lambda_{n,\delta}^{\mathrm{RS}}\to_d(\tilde{M}^\circ - \tilde{m}^\circ)/(\int_{-1}^1 K(s/B^\circ)\Gamma^\circ(s)ds)^{1/2}$, and for λ_n^{VS} under Assumption NS, we have $\lambda_n^{\mathrm{VS}}\sim_p\Lambda_{n,\delta}^{\mathrm{VS}}\to_d\left[\int_0^1\left(\int_0^t U_s^\circ ds\right)^2 dt - \left(\int_0^1\int_0^t U_s^\circ ds dt\right)^2\right]/(\int_{-1}^1 K(s/B^\circ)\Gamma^\circ(s)ds)^{1/2}$. This complete the proof of the corollary. $\square$

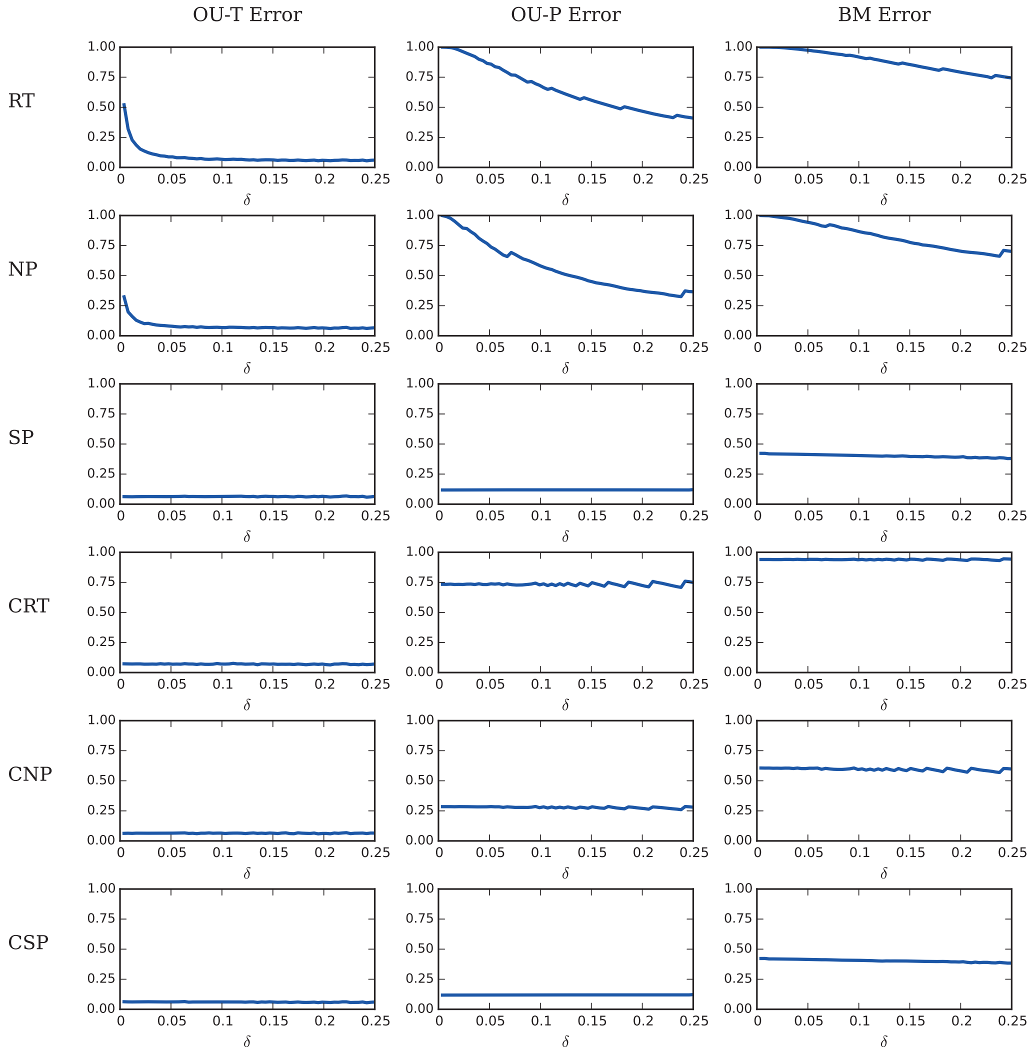
C. Figures for Illustration Simulation

Fig. 3. Rejection Probabilities of 5% RB-KPSS Test for Stationarity



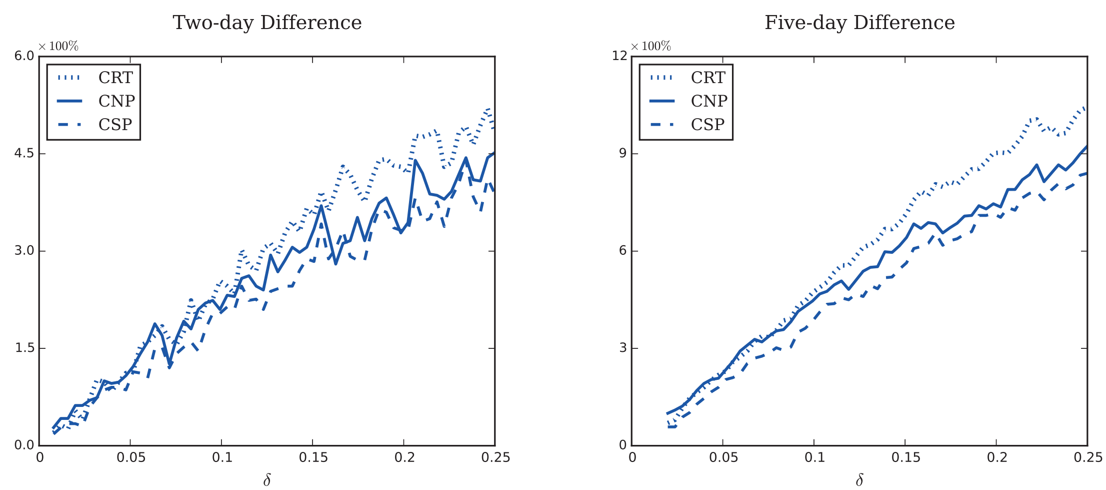
Notes: Presented are the rejection probabilities of the 5% RB-KPSS test for stationarity. Discrete samples are collected at 63 equally-spaced sampling intervals ranging from $\delta = 1/252$ (daily frequency) to $\delta = 1/4$ (quarterly frequency).

Fig. 4. Rejection Probabilities of 5% RB-KPSS Test for Cointegration



Notes: Presented are the rejection probabilities of the 5% RB-KPSS test for cointegration. Discrete samples are collected at 63 equally-spaced sampling intervals ranging from $\delta = 1/252$ (daily frequency) to $\delta = 1/4$ (quarterly frequency).

Fig. 5. Contradictory Rejection Rates



Notes: Presented are the contradictory rejection rates of the 5% RB-KPSS test for stationarity with CRT, CNP and CSP. Discrete samples are collected at 63 equally-spaced sampling intervals ranging from $\delta = 1/252$ (daily frequency) to $\delta = 1/4$ (quarterly frequency). Rejection decisions are made over 5000 simulation iterations, from which contradictory rejection rates are computed as we change the sampling frequency by one day up to two days and five days.