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Optimizing over all of the states of the world together might be difficult even for a machine. This paper adds to the behavioral literature by introducing a model, in which the agent aggregates the states together, even though she is aware of the entire state space. As a result of the state aggregation, the person solves several problems with fewer variables instead of the initial problem with the entire state space. When the person is SEU-maximizer, the decisions are not affected by the way the states get aggregated. In our model people still have subjective priors over states and events, however, they lump some states together in a non-linear way, which leads to different choices. The paper provides axioms for a state aggregation model, discusses identification of the state aggregation from choices in a complete market setting, discusses comparative statics due to changes in the state aggregation and demonstrates how the model explains a number of ambiguity paradoxes.

1 Introduction

Economists usually assume that people have perfect computational abilities and a good understanding of how the world works. However, in reality, optimizing over all of the states of the world together might be difficult even for a machine. The behavioral literature provides good insights into different ways that people might simplify decision-making. This paper adds to the discussion by introducing a model, in which the agent aggregates the states together, even though she is aware of the entire state space. There might many reasons for doing this: a large number of states, states similar to one another, etc. As a result of state-aggregation, the

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person solves several problems with fewer variables instead of the initial problem with the entire state space. When the person is Subjective Expected Utility (SEU)-maximizer, the decisions are not affected by the way the states get aggregated. However, a vast literature (Ellsberg (1961), Kahneman and Tversky (1979), Halevy (2007), etc.) demonstrates that SEU does not describe peoples choices in practice. We offer a model that is a slight deviation from SEU: people still have subjective priors over states and events, however, they lump some states together in a non-linear way. We obtain that various methods of state aggregation lead to different choices. For this reason, as this paper demonstrates, the state-aggregation of each individual might be identified from observed choices.

To illustrate the idea of state aggregation consider the following example. Imagine a person who has just bought a car and needs to choose an insurance plan. Consider three states of the world: an accident with the agent at fault, an accident with someone else at fault, and a natural disaster. An insurance policy consists of three corresponding deductibles. First, suppose that the agent is a person that does not aggregate states. Generally, her choice is a bundle of deductibles such that her consumption is smoothed between all three states. Note that, in this case, there is no state aggregation, and her subjective partition is the whole state space of the world. Next, imagine a person sitting in front of an insurance agent and reading through the book with descriptions of insurance plans and conditions. The book is difficult to read: it is written in a small font and includes too many conditions. To do a thorough choice is too complicated. The person clearly understands the state "accident with the agent at fault." However, states "accident with the other party at fault" and "natural disaster" are not that different from the agent's perspective in both of these cases it is not agent's fault. Thus, in order to simplify her decision process, the person combines collision and natural disaster into event "not my fault." The presence of the state-aggregation changes the way that the agent thinks about the world: Instead of all three states, the state space is partitioned into aggregated events "my fault" and "not my fault." Moreover, the individual generally wants to smooth her consumption in two ways: (1) between her aggregated events, and (2) between the states inside the events. As a result, the choice of deductibles will differ from the insurance plan under no state aggregation, implying that the state aggregation might be identified from choices.

In most economic situations, researchers treat the decision-making process like a black box, and all differences in behavior are usually explained by differences in preferences or information. This paper allows for situations in which agents with the same information and preferences might make different choices due to heterogeneity in the state aggregation. Moreover, such behavior cannot be obtained by providing the agent with some kind of incomplete information, which is present in the model in a form of the subjective probabilities of the states. Incomplete information under no state-aggregation would imply smoothing consumption between all states given some priors. Choices in our model under non-trivial state aggregation do not satisfy consumption-smoothing across all states with any subjective probabilities.

Note that the state aggregation can be interpreted as a frame in Salant and Rubinstein (2008). The authors point it out that "real-life behaviour often depends on observable information, other than the set of feasible alternatives, which is irrelevant in the rational assessment of the alternatives but nonetheless affects behaviour." One of the examples they use is a voter that may be influenced by the order of candidates on a ballot, and they call such additional information a frame. In this paper, an insurance company decides how to formulate terms and conditions: which states should be grouped in which section, the order of the sections, which information should be in the footnote, etc. This kind of information (frame) should be irrelevant to the rational agent, however, in our model, it defines the state aggregation.

In addition, the notion of a state aggregation is very close to intermediate information. Li (2011) introduced a concept of intermediate information – information that arrives after a choice has been made and before an outcome is realized. The author axiomatizes preference relations over pairs of acts and intermediate information together. In order to obtain such preference relations over pairs, Li derives preference relations between acts under some specific information by using conditional preferences and basic, "no information" preferences as primitives. A state aggregation can be interpreted as fixed intermediate information. However, it requires different axiomatization because observable primitives are different.

Undoubtedly, understanding the state aggregation is important for researchers and policy-makers: It is easy to imagine a situation, in which an insurance company might be interested in complicating terms and conditions in order to nudge agents to aggregate states in a desired way. Social planner might want to prevent the insurance company from doing so, for example, by restricting choice set of insurance plans.

In this paper, we (1) offer a model of state aggregation and axioms that define the representation of the preferences; (2) identify the state aggregation from observable choices; (3) provide comparative statics for an insurance company; and, (4) demonstrate how the model explains a number of ambiguity paradoxes.

First, we provide axioms and the representation of the preferences with state aggregation. We assume that only preferences over acts are observed. Thus, our primitive is preferences over acts under some state aggregation from which we derive conditional preferences. We define conditional preferences only for events that we call "aggregating." Such events satisfy a property similar to Savage's Sure Thing Principle. After that, if there is a partition of the whole state space that consists of aggregating events only, we provide a general representation of the preferences. Note that the representation is recursive in its nature: The value on events is defined in the first (conditional) stage, and later the value of the act (ex-ante stage) is a function of the values of the events.

Second, we suggest the simplest non-SEU model of the state aggregation: We assume that the conditional functional that represents the value at each event is a positive non-affine transformation of some SEU over the states in each event. Furthermore, the ex-ante functional is just an expected value of the events. We call this model State Aggregation SEU (SASEU).

Third, we show identification of the unobservable state aggregation, priors and utility from choices and prices in a complete market under SASEU. First, we use the first order conditions to determine the state aggregation. Note that SASEU implies SEU inside of each event, and, hence, produces the same marginal utility ratios when taken between the states inside of the same event. However, when the similar condition is obtain from two states from the different events, it differs from SEU by including additional variables. The last leads to identification of each event, and, thus, the state aggregation. The rest is standard: (1) SEU inside of each event (conditional stage) provides with conditional probabilities and utility up to affine transformation; and, (2) SEU over events (ex-ante stage) delivers probabilities of the events together with the transformation applied to the conditional value.

Fourth, we provide comparative statics on how consumption in each state will change due to changes in the state aggregation. An interesting property holds: whenever something changes in the state aggregation, consumption in all unaffected events moves in the same direction. When an event is being split into two, the direction of change in consumption depends on the current values at the future events. If the current value at one of the events is higher than at another, then consumption smoothing over the states will force the values get closer to each other under the proposed change. Thus, consumption in the states of the event with higher value will fall, while consumption in the states of the event with lower value will increase. On one hand, these implications could be used by an insurance company or portfolio manager to increase their profit. On the other hand, the social planner could use the results to prevent undesirable behaviors.

Finally, SASEU offers an alternative explanation of ambiguity paradoxes suggested in Ellsberg (1961), Machina (2009) and Machina (2014). Halevy (2007) demonstrates empirical evidence of the relationship between non-reduction of compound lotteries and ambiguity aversion. In our model, there is no need for ambiguity or multiple priors to produce the phenomena. The agent has reasonable subjective probabilities of the states. Nevertheless, aggregation of "ambiguity" states together and the curvature in the conditional functional contradicts the reduction of compound lotteries. And that is why, similarly to Dillinberger and Segal (2015), our model is able to explain different ambiguity paradoxes.

Section 2 introduces the model. Section 3 discusses the identification of the state aggregation from choices and prices in the complete market setting. Comparative statics is covered in Section 4. Section 5 demonstrates how SASEU explains ambiguity paradoxes. Axioms and the representation results are in Section 6. Section 7 concludes. All proofs are provided in Appendix.

2 State Aggregation SEU

We start with introducing State Aggregation Subjective Expected Utility (SASEU) model. The value functional of the consumer is defined in a recursive manner and consists of two stages: (1) the conditional stage in which several states are combined into one event and the value of an act is evaluated at the event, and (2) the ex-ante stage in which an act is evaluated across the events of the defined state aggregation.

Definition 1. *The agent's behavior is said to exhibit SASEU Representation if there exist a partition of the state space* π*, probabilities* $P(s|A)$ *and* $P(A|\pi)$ *for any state* $s \in A \in \pi$*, a continuous monotone function* $u : X \to \mathbb{R}$*, and an increasing function* $\phi : \mathbb{R} \to \mathbb{R}$ *such that the agent optimizes the functional* $V(\cdot|\pi)$*:*

$$V(x|A) = \phi\left(\sum_{s\in A} u(x(s))P(s|A)\right)$$
$$V(x|\pi) = \sum_{A\in\pi} V(x|A)P(A).$$

Notice that we assume that $V(\cdot|\pi)$ is a regular SEU functional, while $V(\cdot|A)$ is a positive transformation of a SEU functional in the conditional stage. If $\phi(\cdot)$ is linear, then the agent is a regular SEU-maximizer.

A set of axioms for the above representation is provided in Section 6. Definition 1 implies that the agent evaluates each act given a partition by a folding-back procedure: First, she aggregates states into event A and evaluates the act x given every such event A in the partition π:

$$V(x|A) = \phi\left(\sum_{s\in A} u(x(s))P(s|A)\right),$$

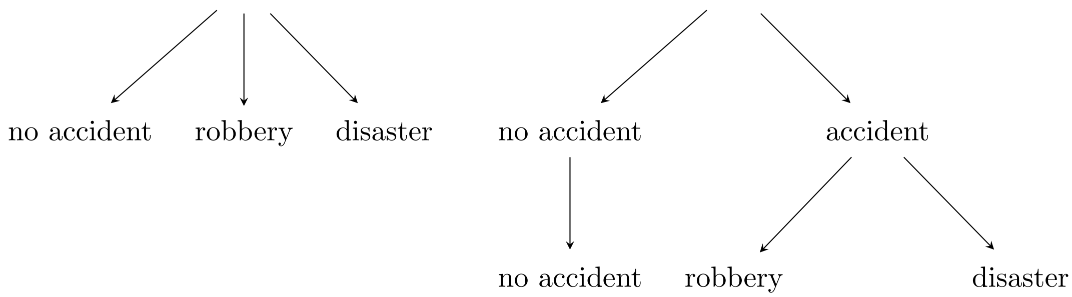
where $P(s|A)$ is the conditional probability of state s given event A. Second, the agent evaluates the act across events that form her state aggregation. Thus, at the ex-ante stage the agent's value of the act is obtained as expected utility with the conditional value functional as a utility:

$$V(x|\pi) = \sum_{A\in\pi} V(x|A)P(A),$$

where $P(A)$ is probability of event A.

Example 1: The agent chooses a home insurance policy under three states of the world: natural disaster, robbery, and no accident.

Suppose that the probability of no accident (s_1) is 90%, the probabilities of robbery (s_2) is 7% and 3% is the probability of natural disaster (s_3). Imagine two different situations: (1) the agent does not aggregate states, and her state space is $\pi_0 = \{s_1, s_2, s_3\}$; and (2) the agent has difficulty optimizing over three states, and she splits the whole outcome space into events "no accident" $A_1 = \{s_1\}$ and "accident" $A_2 = \{s_2, s_3\}$. We denote the state aggregation in this case as $\pi = \{A_1, A_2\}$.



In situation (1), the ex-ante stage implies regular evaluation over the whole state space. Thus, the agent's functional will be

$$V(x|\pi_0) = 0.9\phi(u(x_1)) + 0.07\phi(u(x_2)) + 0.03\phi(u(x_3)),$$

where x_i denotes consumption in state s_i.

In situation (2), the agent's state aggregation π consists of two events, A_1 and A_2. In addition, consider that the agent updates priors using Bayes' rule. Thus, the set of ex-ante priors is $P(A_1) = 0.9$ and $P(A_2) = 0.1$. In the case of no accident, the conditional on event A_1 probability is degenerate: $P(s_1|A_1) = 1$. If the event is "accident," then the conditional on event A_2 probabilities are $P(s_2|A_2) = 0.7$ and $P(s_3|A_2) = 0.3$. Then the value functional will be

$$V(x|\pi) = 0.9\phi(u(x_1)) + 0.1\phi\,(0.7u(x_2) + 0.3u(x_3)\,).$$

As long as $\phi(\cdot)$ is not linear, maximization of $V(x|\pi_0)$ and $V(x|\pi)$ will result in different solutions.

3 Identification in the market

In this section, we show how to identify the subjective partition from choices of Arrow securities and their prices. Even if Arrow assets are not directly available in the market, as long as the market is complete, Arrow prices can always be uniquely recovered.

3.1 Deductibles as Arrow assets

Note that the problem of choosing an insurance plan that contains deductibles can be represented as a choice among Arrow securities. In order to see this, consider the following example.

Example 2: Suppose that an agent needs to choose an insurance plan while considering three states of the world, as before: robbery with losses L_1; a natural disaster with losses L_2; and no accident, which implies no loss. The price of insurance consists of the sum of prices of chosen deductibles in both states of occurrence. We assume that price $q(d)$ of each deductible d is linear in its amount d: $q(d) = a - p_d d$, where a is a constant state premium. Thus, we can define the price of a \$1 deductible decrease as $p_d = q(d) - q(d+1)$, which is constant for any amount d due to assumed linearity. Notice that p_d is the price of a corresponding Arrow security: decreasing the deductible by \$1 in some state is equivalent to increasing consumption by \$1 in the same state.

We denote the deductible in the case of robbery as d_1, the price of \$1 of deductible in this state as p_{d_1}, and a_1 as the state premium. Similarly, d_2, p_{d_2}, and a_2 are the deductible, the \$1 price in the case of the natural disaster, and the state premium, respectively. Thus, the total price of the insurance plan (d_1, d_2) is $p = (L_1 - d_1)p_{d_1} + a_1 + (L_2 - d_2)p_{d_2} + a_2$. The agent has income I and optimizes consumption $c = (c_1, c_2, c_3)$ in all states of the world given some state aggregation π. All the money left (s) after paying for insurance is used for consumption during the year. Thus, the agent's problem is

$$V(c|\pi) \to \max$$

$$\text{s.t. } (L_1 - d_1)p_{d_1} + (L_2 - d_2)p_{d_2} + a_1 + a_2 + s = I$$

$$c_1 = s - d_1;\; c_2 = s - d_2;\; c_3 = s.$$

We now rewrite the problem in terms of Arrow securities. Notice that consumption bundle $c = (c_1, c_2, c_3)$ is exactly the portfolio of Arrow assets that the agent chooses. The only thing left to do is to find Arrow prices (p_1, p_2, p_3) and rewrite the budget constraint in an appropriate form. Denote $\tilde{I} = I - L_1 p_1 - L_2 p_2 - a_1 - a_2$ the agent's total endowment when taking potential losses and all state premiums into account. Also, as mentioned above, $p_1 = p_{d_1}$ and $p_2 = p_{d_2}$. Then, we can rewrite the budget constraint as

$$-d_1 p_{d_1} - d_2 p_{d_2} + s = (c_1 - s)p_1 + (c_2 - s)p_2 + s = c_1 p_1 + c_2 p_2 + c_3(1 - p_1 - p_2) = \tilde{I}.$$

Thus, $p_3 = 1 - p_1 - p_2$ and the agent's problem is

$$V(c|\pi) \to \max$$

$$\text{s.t. } c_1 p_1 + c_2 p_2 + c_3 p_3 = \tilde{I}.$$

As a consequence, if we observe the choices of deductibles with their prices and the agent's income, we can recover the amount of Arrow securities/consumption in each state.

3.2 Non-parametric identification example

Suppose that $\Omega = \{s_1, s_2, s_3\}$, partition $\pi = \{A_1, A_2\}$, where $A_1 = \{s_1\}$ and $A_2 = \{s_2, s_3\}$, and priors are such that $p_{A_1} + p_{A_2} = 1$, $P(s_1|A_1) = 1$ and $P(s_2|A_2) + P(s_3|A_2) = 1$. Consider that a complete set of Arrow securities is available on the market.[1] The agent purchases a bundle of Arrow securities that maximizes her value given a certain amount of income I and the price p_i of an Arrow security that pays 1 in state i:

$$V((x_1, x_2, x_3)|\pi) \to \max_x$$
$$\text{s.t. } p_1 x_1 + p_2 x_2 + p_3 x_3 = I.$$

Choices $x = (x_1, x_2, x_3)$, prices $p = (p_1, p_2, p_3)$ and income I are observed. We assume that all possible combinations of (p, I) are available. The purpose is to identify the state aggregation π, utility function $u(\cdot)$, transformation function $\phi(\cdot)$, and priors p_{A_1}, p_{A_2}, $P(s_2|A_2)$, $P(s_3|A_2)$.

[1] An Arrow security pays one unit in a specified state and zero otherwise.

The agents solves her maximization problem and chooses some bundle x:

$$p_{A_1}\phi(u(x_1)) + p_{A_2}\phi\left(P(s_2|A_2)u(x_2) + P(s_3|A_2)u(x_3)\right) \to \max_x$$
$$\text{s.t. } p_1^x x_1 + p_2^x x_2 + p_3^x x_3 = I.$$

Hence, if λ is a Lagrange multiplier, then the first order conditions are

$$p_{A_1}\phi'(u(x_1))u'(x_1) = \lambda p_1^x$$
$$p_{A_2}\phi'\big(P(s_2|A_2)u(x_2) + P(s_3|A_2)u(x_3))p(s_2|A_2)u'(x_2)\big) = \lambda p_2^x$$
$$p_{A_2}\phi'\big(P(s_2|A_2)u(x_2) + P(s_3|A_2)u(x_3))p(s_3|A_2)u'(x_3)\big) = \lambda p_3^x.$$

Two states from the same event: If we pick s_2 and s_3, we obtain

$$\frac{p(s_2|A_2)}{p(s_3|A_2)}\frac{u'(x_2)}{u'(x_3)} = \frac{p_2^x}{p_3^x}. \tag{1}$$

Now choose some other bundle $y = (y_1, y_2, y_3)$ such that $y_2 = x_2$, but $y_3 \neq x_3$. Then we observe similar first order conditions:

$$\frac{p(s_2|A_2)}{p(s_3|A_2)}\frac{u'(y_2)}{u'(y_3)} = \frac{p_2^y}{p_3^y}. \tag{2}$$

After dividing (1) by (2), we obtain

$$\frac{u'(y_3)}{u'(x_3)} = \frac{p_2^x}{p_3^x}\frac{p_3^y}{p_2^y}. \tag{3}$$

Two states from different events: We repeat the above derivation for states s_1 and s_2:

$$\frac{p_{A_1}}{p_{A_2}p(s_2|A_2)}\frac{\phi'(u(x_1))u'(x_1)}{\phi'\left(P(s_2|A_2)u(x_2) + P(s_3|A_2)u(x_3))\right)u'(x_2)} = \frac{p_1^x}{p_2^x}. \tag{4}$$

Choose another bundle $z = (z_1, z_2, z_3)$ such that $z_1 = x_1$, but $z_2 \neq x_2$. After dividing (4) by the corresponding first order condition for bundle z, we get

$$\frac{\phi'\left(P(s_2|A_2)u(z_2)+P(s_3|A_2)u(z_3)\right)}{\phi'\left(P(s_2|A_2)u(x_2)+P(s_3|A_2)u(x_3)\right)}\frac{u'(z_2)}{u'(x_2)}=\frac{p_1^x}{p_2^x}\frac{p_2^z}{p_1^z}. \tag{5}$$

Compare now (3) and (5). In (3), the left side does not depend on values of x_1, y_1 and $x_2 = y_2$. Thus, if we pick other bundles with the same x_3 and y_3, we will obtain the same value of the ratio in (3). In (5), the left side depends on x_3 and z_3. So as long as $\phi(\cdot)$ is not a linear function, changing x_3 and z_3 while keeping everything else constant will result in different values on the right side of (5). Hence, we know that the state aggregation is $\pi = \{\{s_1\}, \{s_2, s_3\}\}$.

Now note that by choosing different values of x_3 and y_3 in (3), we can identify $u(\cdot)$ up to affine transformation. After that we consider (1) again and identify the probability ratio $\frac{P(s_2|A_2)}{P(s_3|A_2)}$. The last means that we can identify the probabilities themselves:

$$\frac{P(s_2|A_2)}{P(s_3|A_2)}=c \text{ and } P(s_2|A_2)+P(s_3|A_2)=1 \Rightarrow P(s_2|A_2)=\frac{c}{c+1} \text{ and } P(s_3|A_2)=\frac{1}{c+1}.$$

In order to identify $\phi(\cdot)$, we consider (5) again. $u(\cdot)$, $P(s_2|A_2)$ and $P(s_3|A_2)$ have already been identified, hence, we can obtain the value of the ratio $\frac{\phi'(P(s_2|A_2)u(z_2)+P(s_3|A_2)u(z_3)))}{\phi'(P(s_2|A_2)u(x_2)+P(s_3|A_2)u(x_3)))}$. By choosing different x_2, x_3, z_2 and z_3, we identify $\phi(\cdot)$ up to affine transformation.

The only unknown variables left are p_{A_1} and p_{A_2}. However, we obtain $\frac{p_{A_1}}{p_{A_2}}$ from (4). Finally, given that $p_{A_1} + p_{A_2} = 1$, we identify p_{A_1} and p_{A_2}.

3.3 Identification

Thus, the agent wants to buy a portfolio of securities and aggregates states of the world into some partition. The purpose of this section is to identify the agent's state aggregation and priors from choices of Arrow assets and their prices. In order to do so, we generalize the above non-parametric example; however, the idea behind the method stays the same.

In order to achieve convexity of indifference curves and have the optimality condition hold in each region, we use the following technical assumption:

Assumption 1. *$u(\cdot)$ and $\phi(\cdot)$ are strictly increasing twice-differentiable functions defined on a compact support and such that $\phi\left(\sum_{s\in A} q_s u(x_s)\right)$ is concave for any $A \subseteq \Omega$ and any probability* *measure q such that $\sum_{s\in A} q_s = 1$.*

Assumption 2 guarantees that the agent is not a SEU-maximizer unless the agent's state aggregation is the whole state space, i.e. no state aggregation. It is crucial for identification because different partitions do not affect choices if the individual behaves according to SEU.

Assumption 2. *$\phi(\cdot)$ is not a linear function.*

Assumption 3 forbids aggregation of all states together. It is necessary because the agent's behavior under no aggregation and under full aggregation would follow SEU, making it impossible to identify the exact situation without any additional information. [2]

Assumption 3. *π is not equal to $\{\Omega\}$.*

Theorem 1. *If Assumptions 1–3 hold, then the state aggregation π and probabilities p_A and $P(s|A)$ are identified for any $s \in A \in \pi$. In addition, functions $u(\cdot)$ and $\phi(\cdot)$ are identified up to affine transformation.*

The proof of the theorem follows the identification procedure shown in the previous subsection.

4 SASEU and insurance

In this section, we discuss how changes in the state aggregation might affect consumption.

Suppose there are S states of the world $\Omega = \{s_1, \ldots, s_S\}$. Consider an agent who chooses consumption $c = (c_1, \ldots, c_S)$ for every state of the world. Income is I, the price of 1 unit of consumption in state s is p_s, and the state probabilities are objective and denote them by $P(s)$.

Consider first an agent who does not aggregate states. Then the problem she solves is

$$\sum_{s\in\Omega} P(s)\phi(u(c_s)) \to \max_{c_1,\ldots,c_S}$$
$$\text{s.t. } \sum_{s\in\Omega} p_s c_s = I.$$

[2] As long as $\phi(\cdot)$ is not linear, Ω and $\{\Omega\}$ will produce different choices. Both will be consistent with SEU, however, with different utilities.

For any two states $s_1, s_2 \in \Omega$, the first order condition is

$$\frac{p_{s_1}}{p_{s_2}} = \frac{P(s_1)}{P(s_2)} \frac{\phi'(u(c_{s_1}))}{\phi'(u(c_{s_2}))} \frac{u'(c_{s_1})}{u'(c_{s_2})}. \tag{6}$$

Now suppose that the agent aggregates states into π and updates probabilities according to the definition of conditional probability, i.e., if $s \in A$ then $P(s) = P(A)P(s|A)$. Hence, the agent's problem is

$$\sum_{A \in \pi} P(A) \phi \left(\sum_{s \in A} P(s|A) u(c_s) \right) \to \max_{c_1, \dots, c_S}$$
$$\text{s.t.} \sum_{s \in \Omega} p_s c_s = I.$$

If $s_1 \in A_1$ and $s_2 \in A_2$, then the first order condition is

$$\frac{p_{s_1}}{p_{s_2}} = \frac{P(s_1)}{P(s_2)} \frac{\phi'\left(\sum_{s \in A_1} P(s|A_1) u(c_s)\right)}{\phi'\left(\sum_{s \in A_2} P(s|A_2) u(c_s)\right)} \frac{u'(c_{s_1})}{u'(c_{s_2})}. \tag{7}$$

Note that $\frac{u'(c_{s_1})}{u'(c_{s_2})}$ is present in both equations and is responsible for consumption smoothing due to risk-aversion. However, $\frac{\phi'\left(\sum_{s \in A_1} P(s|A_1) u(c_s)\right)}{\phi'\left(\sum_{s \in A_2} P(s|A_2) u(c_s)\right)}$ is different. If $\phi(\cdot)$ is concave then the ratio produces consumption smoothing over events due to risk-aversion over the events. On the other hand, if $\phi(\cdot)$ is convex then the agent is risk-loving towards the events. Thus, in combination with consumption smoothing over states that comes from $u(\cdot)$, the state aggregation might have different effects on choices.

The next theorem provides comparative statics in the situation, when one of the events is split into two in the state aggregation. Note that combining two events is also included, and it will produce an opposite result.

Theorem 2. *Consider two partitions π and $\tilde{\pi}$ such that $\tilde{\pi} = \{A_1 \backslash B, B, A_2, \dots, A_k\}$ and $\pi = \{A_1, A_2, \dots, A_k\}$. In addition, suppose that both $u(\cdot)$ and $\phi(\cdot)$ are concave and differentiable. Then*

1. $V_B(c) > V_{A_1 \backslash B}(c) \Leftrightarrow V_B(\tilde{c}) > V_{A_1 \backslash B}(\tilde{c}) \Leftrightarrow \tilde{c}_j < c_j, \tilde{c}_i > c_i$ *for any* $s_i \in A_1 \backslash B$ *and* $s_j \in B$;

2. $V_B(c) < V_{A_1 \backslash B}(c) \Leftrightarrow V_B(\tilde{c}) < V_{A_1 \backslash B}(\tilde{c}) \Leftrightarrow \tilde{c}_j > c_j, \tilde{c}_i < c_i$ *for any* $s_i \in A_1 \backslash B$ *and* $s_j \in B$*;*

3. $V_B(c) = V_{A_1 \backslash B}(c) \Leftrightarrow V_B(\tilde{c}) = V_{A_1 \backslash B}(\tilde{c}) \Leftrightarrow \tilde{c}_j = c_j, \tilde{c}_i = c_i$ *for any* $s_i \in A_1 \backslash B$ *and* $s_j \in B$*.*

Moreover, one of the following holds:

1. $\tilde{c}_s > c_s$ *for all* $s \in A_k$ *and all* $k \neq 1$*;*

2. $\tilde{c}_s < c_s$ *for all* $s \in A_k$ *and all* $k \neq 1$*;*

3. $\tilde{c}_s = c_s$ *for all* $s \in A_k$ *and all* $k \neq 1$*.*

The above theorem predicts the following: First, if a state does not belong to the affected event, then the consumption in this state will change in the same direction as consumption in other unaffected states and events. Second, if an event gets split into two, $A_1 \backslash B$ and B, and the current value at one of them is greater (e.g., $V_B(c) > V_{A \backslash B}(c)$), then consumption smoothing over the events will push the values towards each other, making consumption in the states of the event with the higher value (B) decrease, and the consumption in the states of the event with the lower value ($A_1 \backslash B$) increase. Third, if two events get aggregated, then the effect is opposite. The pressure that pushed the values closer due to consumption smoothing from events is gone. Hence, the values at these events will pull away from each other. The last means that the consumption in the states of the event with greater value will increase even more and the consumption in the states of the event with the lower value will decrease.

The above theorems applies to the insurance market as it was discussed in the previous section. It is not unreasonable to imagine a situation in which an insurance company observes current state-aggregation, probabilities, state prices and consumption. If the company is able to manipulate the state aggregation, it will be interested in predicting corresponding changes in deductibles and/or consumption. It would not be a difficult task if utilities were known, however, this is a unrealistic assumption. Even if the company observed choices of the consumers for 10 years, given heterogenous consumers, it would be still impossible to evaluate utility function even if the preferences are stable over time. The above theorem attempts to give some predictions in this situation. Unfortunately, it relies on the comparison of current values at the events ($V_B(c)$ vs. $V_{A \backslash B}(c)$), which still depends on the utility. However, in some situations it might be the case that all possible utility functions that could generate the observable consumption imply the same relation between the values at the events. Consider the following example.

Example 3: Suppose that the insurance company is interested in splitting event $A = \{s_1, s_2, s_3, s_4\}$ into events $B_1 = \{s_1, s_4\}$ and $B_2 = \{s_2, s_3\}$. The probabilities are $P(s_1|B_1) = 0.3$ and $P(s_4|B_1) = 0.7$, while $P(s_2|B_2) = P(s_3|B_2) = 0.5$ and $P(B_1) = P(B_2)$. The current consumption is $c_1 = 2$, $c_2 = 4$, $c_3 = 7$ and $c_4 = 9$ and the state prices are $p_1 = 1.8$, $p_2 = 2.5$, $p_3 = 1.5$ and $p_4 = 1$. Note that by the Taylor expansion, we obtain the following.

$$u(c_2) = u(c_1) + u'(t_1)(c_2 - c_1), \text{ where } t_1 \in [c_1, c_2]$$
$$u(c_3) = u(c_2) + u'(t_2)(c_3 - c_2), \text{ where } t_2 \in [c_2, c_3]$$
$$u(c_4) = u(c_3) + u'(t_3)(c_4 - c_3), \text{ where } t_3 \in [c_3, c_4]$$

Hence, the values at the events are

$$\begin{aligned} V_{B_1}(c) &= 0.3u(c_1) + 0.7u(c_4) = u(c_1) + 0.7u'(t_3)(c_4 - c_3) + 0.7u'(t_2)(c_3 - c_2) + 0.7u'(t_1)(c_2 - c_1) \\ &= u(c_1) + 1.4u'(t_3) + 2.1u'(t_2) + 1.4u'(t_1) \\ V_{B_2}(c) &= 0.5u(c_2) + 0.5u(c_3) = u(c_1) + u'(t_1)(c_2 - c_1) + 0.5u'(t_2)(c_3 - c_2) \\ &= u(c_1) + 2u'(t_1) + 1.5u'(t_2) \end{aligned}$$

Thus, $V_{B_1}(c) > V_{B_2}(c)$ if and only if $1.4u'(t_3) + 0.6u'(t_2) - 0.6u'(t_1) > 0$. Note that it depends only on the values of the utility derivatives, for which we can establish bounds from the first order conditions. Inside of the event A, the FOC implies $\frac{u'(c_k)}{u'(c_j)} = \frac{p_k}{p_j}\frac{P(s_j)}{P(s_k)}$. Hence, we get the following.

$$\frac{u'(c_1)}{u'(c_4)} = 4.2; \ \frac{u'(c_2)}{u'(c_4)} = 3.5; \ \frac{u'(c_3)}{u'(c_4)} = 2.1$$

For simplicity, we normalize $u'(c_4) = 1$, then the bounds for the expression of the interest are

$$\max(1.4u'(t_3) + 0.6u'(t_2) - 0.6u'(t_1)) = 2.94 > 0$$
$$\min(1.4u'(t_3) + 0.6u'(t_2) - 0.6u'(t_1)) = 0.14 > 0$$

Thus, we conclude that $V_{B_1}(c) > V_{B_2}(c)$ and can predict that splitting event A into B_1 and B_2 will result in decrease of consumption in states s_1 and s_4, and increase of consumption in states s_2 and s_3. One can easily generalize this approach.

5 Ambiguity Paradoxes

In this section, we go over well-known thought experiments that demonstrate problems that the most prominent theories have in explaining behavior. Dillinberger and Segal (2015) explain the paradoxes with the recursive nonexpected utility model of Segal (1987). The model assumes that the agent has a distribution of priors over possible probability distributions and also uses nonexpected utility functional for evaluation of each possibility. SASEU offers an alternative explanation.

Ellsberg paradox

Consider the following famous paradox from Ellsberg (1961): there is an urn that contains 90 colored balls. 30 balls are red, all other balls are black and yellow, but it is unknown in which proportion. One ball is randomly picked from the urn and four lotteries are considered:

$$A_1 : \text{ 100\$ if the ball is red}$$

$$A_2 : \text{ 100\$ if the ball is black}$$

$$B_1 : \text{ 100\$ if the ball is red or yellow}$$

$$B_2 : \text{ 100\$ if the ball is black or yellow}$$

The subjects are offered to choose separately between A_1 and A_2, and between B_1 and B_2. It is a well-established fact that the majority of people prefer $A_1 \succ A_2$ and $B_2 \succ B_1$, which contradicts the classical SEU. Many models were proposed to explain the paradox. We will add one more to the list:

The state space consists of the states red (R), black (B) and yellow (Y), i.e., $\Omega = \{R, B, Y\}$. The agent might be naturally inclined to aggregate states B and Y into one event that we will call BY. Thus, obtaining the state aggregation $\pi = \{R, BY\}$. The probabilities of the events in

π are objective: $P(R) = \frac{1}{3}$ and $P(BY) = \frac{2}{3}$. The probabilities of the states in BY are subjective, however, there is no reason to believe that $P(B|BY) \neq P(Y|BY)$ due to the symmetry of the situation. Hence, $P(B|BY) = P(Y|BY) = 0.5$. Let us demonstrate how a SASEU-maximizer evaluates the above lotteries in this case:

$$V(A_1) = \frac{1}{3}\phi\left(u(100)\right) + \frac{2}{3}\phi\left(u(0)\right)$$
$$V(A_2) = \frac{1}{3}\phi\left(u(0)\right) + \frac{2}{3}\phi\left(0.5u(100) + 0.5u(0)\right).$$

Notice that $V(A_1) > V(A_2)$ if and only if

$$0.5\phi(u(100)) + 0.5\phi(u(0)) > \phi\left(0.5u(100) + 0.5u(0)\right). \tag{8}$$

The other two lotteries:

$$V(B_1) = \frac{1}{3}\phi\left(u(100)\right) + \frac{2}{3}\phi\left(0.5u(100) + 0.5u(0)\right)$$
$$V(B_2) = \frac{1}{3}\phi\left(u(0)\right) + \frac{2}{3}\phi\left(u(100)\right).$$

Moreover, $V(B_2) > V(B_1)$ if and only if condition Equation (8) holds! The condition is trivially satisfied if $\phi(\cdot)$ is concave.

Slightly-Bent Coin problem

In Machina (2014), the author offers the following thought experiment: an agent needs to choose between two bets. The payout of the subject depends on a flip of a slightly-bent coin (it is unknown in which direction the coin is bent) and the color of the ball drawn from the urn that contains two balls, each of them can be black or white. Hence, the state space consists of four states dependent on whether the ball is black or white and whether the coin lands heads or tail, i.e., $\Omega = \{BH, BT, WH, WT\}$. The bets are as follows.

$$\text{Bet I: } 8,000\$ \text{ if } BH; \ -8,000\$ \text{ if } BT$$

$$\text{Bet II: } 8,000\$ \text{ if } WT; \ -8,000\$ \text{ if } BT$$

Machina argues that Choquet expected utility predicts indifference in this case, while real people might have strong preferences for one of the bets.

Consider a SASEU-maximizer that aggregates states with different balls and the same coin together. Thus, states BH and WH are aggregated into event H, and states BT and WT are aggregated into event T, forming a subjective partition $\pi = \{H, T\}$. First of all, there is no reason to believe that probabilities of white and black balls differ, so $P(W|H) = P(B|H) = P(W|T) = P(B|T) = 0.5$. In the same manner, there is no reason to believe that the coin is bent in specific direction and probabilities of heads and tail differ, implying $P(H) = P(T) = 0.5$. The value of each bet can be calculated as

$$V(I) = 0.5\phi\left(0.5u(8,000) + 0.5u(-8,000)\right) + 0.5\phi(u(0))$$
$$V(II) = 0.5\phi\left(0.5u(0) + 0.5u(-8,000)\right) + 0.5\phi\left(0.5u(0) + 0.5u(8,000)\right).$$

Notice that as long as $\phi(\cdot)$ is not linear, the choice between Bet I and Bet II depends on the exact values of the functions $\phi(\cdot)$ and $u(\cdot)$. For example, if $\phi(x) = \sqrt{x}$, $u(-8,000) = 0$, $u(0) = 2$ and $u(8,000) = 4$, then $II \succ I$. However, if we change only $u(8,000) = 3$ and keeping the rest the same, we will observe that $I \succ II$[3].

Ambiguity at Low vs. High Outcomes problem

This paradox was also proposed in Machina (2014). The subject is asked to choose between two urns. Both urns contain three balls, one of them is known to be red. Each of the other balls can be either black or white. The value of c is defined as certainty equivalent of 50:50 bet for 0\$ and 100\$. The payoffs of the urns are shown in the Table.

Table 1: Ambiguity at Low vs. High Outcomes

Urn	R	B	W
I	100\$	0\$	c\$
II	0\$	c\$	100\$

[3] In the latter case, $u(\cdot)$ is concave and together with concave $\phi(\cdot)$ it generates preferences consistent with Machina's expectation about the ambiguity averse decision-maker.

The most prominent ambiguity theories (MEU, Choquet, Smooth ambiguity and Variational preferences) predict indifference between the urns. However, Machina argues that the subjects might have strong preference towards one of the urns.

SASEU agent might naturally want to aggregate states B and W into event A for both urns. Then probabilities of the events R and A are objective: $P(R) = \frac{1}{3}$ and $P(A) = \frac{2}{3}$. In addition, given event A, there is no reason to believe that probabilities of B and W are different for any of the urns, thus, $P(B|A) = P(W|A) = 0.5$. Hence, the value of each urn is as follows.

$$V(I) = \frac{1}{3}\phi\left(u(100)\right) + \frac{2}{3}\phi\left(0.5u(c) + 0.5u(0)\right) = \frac{1}{3}\phi\left(u(100)\right) + \frac{2}{3}\phi\left(0.25u(100) + 0.75u(0)\right)$$
$$V(II) = \frac{1}{3}\phi\left(u(0)\right) + \frac{2}{3}\phi\left(0.5u(c) + 0.5u(100)\right) = \frac{1}{3}\phi\left(u(0)\right) + \frac{2}{3}\phi\left(0.75u(100) + 0.25u(0)\right).$$

Notice that as long as $\phi(\cdot)$ is not linear, dependent on $\phi(\cdot)$ and $u(\cdot)$, any behavior might be obtained. For example, if $\phi(x) = \sqrt{x}$, $u(0) = 0$ and $u(100) = 1$, then $I \succ II$.

50:51 Example

This paradox is described in Machina (2009). An urn contains 101 balls, 50 of which are marked either with 1 or 2, and 51 balls are marked either with 3 or 4. One ball is drawn at random. The subject is offered to choose between lotteries f_1 and f_2, and between f_3 and f_4, payoffs for which are shown in the table.

Table 2: 50:51 Example

Lottery	E_1	E_2	E_3	E_4
f_1	8,000$	8,000$	4,000$	4,000$
f_2	8,000$	4,000$	8,000$	4,000$
f_3	12,000$	8,000$	4,000$	0$
f_4	12,000$	4,000$	8,000$	0$

Lotteries f_3 and f_4 are obtained from f_1 and f_2 by shifting 4,000$ and adding them to 8,000$. Tail-separability in the Choquet model implies that $f_1 \succ f_2$ if and only if $f_3 \succ f_4$. Machina argues that there is no reason for some subjects not to show preference reversal in this case.

A SASEU-maximizer will aggregate states E_1 and E_2 into event E_{12} and states E_3 and E_4

into event E_{34}. There is no reason for probabilities of E_1 and E_2 given event E_{12} to be different, so $P(E_1|E_{12}) = P(E_2|E_{12}) = 0.5$. By analogy, $P(E_3|E_{34}) = P(E_4|E_{34}) = 0.5$. Probabilities of E_{12} and E_{34} are objective and equal $\frac{50}{101}$ and $\frac{51}{101}$ correspondingly. Thus, the values of the lotteries are

$$V(f_1) = \frac{50}{101}\phi\left(u(8,000)\right) + \frac{51}{101}\phi\left(u(4,000)\right)$$
$$V(f_2) = \phi\left(0.5u(8,000) + 0.5u(4,000)\right)$$
$$V(f_3) = \frac{50}{101}\phi\left(0.5u(12,000) + 0.5u(8,000)\right) + \frac{51}{101}\phi\left(0.5u(4,000) + 0.5u(0)\right)$$
$$V(f_4) = \frac{50}{101}\phi\left(0.5u(12,000) + 0.5u(4,000)\right) + \frac{51}{101}\phi\left(0.5u(8,000) + 0.5u(0)\right).$$

Constructing a preference reversal example in this case is not as trivial as in other paradoxes, however, it is not impossible. Suppose that $u(0) = 0$, $u(4,000) = 0.5$, $u(8,000) = 1$, $u(12,000) = 2$ and $\phi(x) = \begin{cases} x, \text{ if } x \leq 1 \\ x^2, \text{ if } x > 1 \end{cases}$. Then we obtain that $f_2 \succ f_1$ and $f_3 \succ f_4$.

Reflection Example

This thought experiment was also introduced in Machina (2009). An urn contains 100 balls, half of which is marked either with 1 or 2, and another half is marked either with 3 or 4. One ball is drawn at random. The subject is offered to choose between lotteries f_5 and f_6, and between f_7 and f_8, payoffs for which are shown in the table.

Table 3: Reflection Example

Lottery	E_1	E_2	E_3	E_4
f_5	4,000$	8,000$	4,000$	0
f_6	4,000$	4,000$	8,000$	0
f_7	0	8,000$	4,000$	4,000$
f_8	0	4,000$	8,000$	4,000$

Choquet model implies that $f_5 \succ f_6$ if and only if $f_7 \succ f_8$, because f_7 and f_8 are obtained from f_5 and f_6 by switching 0 and 4,000$. Machina argues that there is no difference between lotteries f_5 and f_8, and between f_6 and f_7, implying that $f_5 \succ f_6$ if and only if $f_8 \succ f_7$. Thus, only indifference between four lotteries would be possible.

A SASEU-maximizer will aggregate states E_1 and E_2 into event E_{12} and states E_3 and E_4 into event E_{34}. There is no reason for probabilities of E_1 and E_2 given event E_{12} to be different, so $P(E_1|E_{12}) = P(E_2|E_{12}) = 0.5$. By analogy, $P(E_3|E_{34}) = P(E_4|E_{34}) = 0.5$. Probabilities of E_{12} and E_{34} are objective and equal 0.5. Thus, the values of the lotteries are

$$V(f_5) = V(f_8) = 0.5\phi\left(0.5u(4,000) + 0.5u(8,000)\right) + 0.5\phi\left(0.5u(4,000) + 0.5u(0)\right)$$
$$V(f_6) = V(f_7) = 0.5\phi\left(u(4,000)\right) + 0.5\phi\left(0.5u(8,000) + 0.5u(0)\right).$$

Notice that as long as $\phi(\cdot)$ is not linear, the indifference is not the only option. For example, if $\phi(x) = \sqrt{x}$, $u(0) = 0$, $u(4,000) = 4$ and $u(8,000) = 8$, then $f_5 \succ f_6$. Moreover, $f_5 \succ f_6$ if and only if $f_8 \succ f_7$.

6 Axiomatization

6.1 Preliminaries

Suppose that X is a convex subset of consequences in $\mathbb{R}$, and Ω is a finite set of states of the world with an algebra Σ of subsets of Ω. We denote $\mathcal{F}$ a set of all acts, Σ-measurable finite step functions: $\Omega \to \Delta X$. Let $\mathcal{M}$ be a set of elements π, such that $\pi \subset \Sigma$ is a partition of Ω. Partitions of the state space represent different ways of state aggregation.

We denote for all $f, g \in \mathcal{F}, A \in \Sigma$, fAg an act: $fAg(s) = f(s)$ if $s \in A$, and $fAg(s) = g(s)$ if $s \notin A$. The mixture of acts is defined statewise. We will abuse notation and define X as a set of constant acts in what follows below.

6.2 Axioms and representation

The purpose of this section is to provide axioms of preference relation $\succeq$ between acts f over $\mathcal{F}$ that can be represented by the model of state aggregation. We take as a primitive a preference relation between acts. After that, we induce conditional preferences whenever we can guarantee their completeness.

First, we define the standard set of axioms. Notice that the independence axiom holds only for constant acts.

Axiom 1. *For all $f, g, h \in \mathcal{F}$ and $x, y, z \in X$: (i) $\succeq$ is complete and transitive; (ii) if $\lambda \in (0,1]$: $x \succeq y \Leftrightarrow \lambda x + (1-\lambda)z \succeq \lambda y + (1-\lambda)z$; (iii) if $f \succ g$, and $g \succ h$, then there exist $\lambda, \mu \in (0,1)$ such that $\lambda f + (1-\lambda)h \succ g$ and $g \succ \mu f + (1-\mu)h$; (iv) if $f(s) \succeq g(s)$ for all $s \in \Omega$, then $f \succeq g$; and (v) $\succeq$ is not degenerate.*

In order to introduce state aggregation and obtain complete conditional preferences, we define a concept of an aggregating event:

Definition 2. *Event A is called aggregating if for any $f, g, h, h' \in \mathcal{F}$: $fAh \succeq gAh \Leftrightarrow fAh' \succeq gAh'$.*

An aggregating event satisfies a property similar to Savage's Sure Thing Principle. The independence axiom does not hold in this model, so it is not equivalent to the existence of a unique probability. The aggregating event implies that the value of the act at the event is the only aspect that matters in act evaluation, and not each state value separately.

Now, for all aggregating events A, we define conditional preferences $\succeq_A$:

Definition 3. *For all $f, g, h \in \mathcal{F}$: $f \succeq_A g \Leftrightarrow fAh \succeq gAh$.*

Note that conditional preferences $\succeq_A$ are complete and satisfy all analogous axioms from Axiom 1.

Axiom 2 (State Aggregation - SA)**.** *There exists a partition π of Ω such that for any $A \in \pi$, A is aggregating event.*

Note that $\{\Omega\}$ (all states are aggregated) trivially is aggregating event, and, thus, there always exists a trivial partition. In addition, Ω (all states are separate events) is another trivial partition due to monotonicity.

Theorem 3. *(Representation Theorem) A binary relation $\succeq$ satisfies axioms 1 and 2 if and only if there exist a unique up to affine transformation nonconstant continuous monotone function $u : X \to \mathbb{R}$, monotone continuous functionals $I : \mathbb{R}^{|\pi|} \to \mathbb{R}$ and $I_{A_i} : \mathbb{R}^{|A_i|} \to \mathbb{R}$ for each $A_i \in \pi$*

such that $\succeq_A$ is represented by the unique preference functional $V(\cdot|A) : \mathcal{F} \to \mathbb{R}$, and $\succeq$ is represented by unique $V(\cdot|\pi) : \mathcal{F} \to \mathbb{R}$ such that

$$V(f|\pi) = I(V(f|A_1), V(f|A_2), \ldots, V(f|A_n))$$
$$V(f|A_i) = I_{A_i}(u(f)).$$

Proof. **Sufficiency:** $\succeq$ is continuous and independent preference relation on a mixture space X, thus, by the Mixture Space Theorem, it can be represented by a continuous and monotone utility function $u : X \to \mathbb{R}$. Note that the same applies to $\succeq_A$ on X.

Let $\succeq_A^*$ be preference relation on $u(X)$: $u(f) \succeq_A^* u(g)$ if and only if $f \succeq_A g$. $\succeq_A^*$ is continuous, independent and monotone preference relation on $u(X)$, hence, there exists continuous and monotone $I_A : u(X) \to \mathbb{R}$ that represents $\succeq_A^*$. Thus, define $V(f|A) = I_A(u(f))$.

Let $\succeq_\pi$ be preference relation such that

$$V(f|A_1)A_1V(f|A_2)A_2...V(f|A_n) \succeq_\pi V(g|A_1)A_1V(g|A_2)A_2...V(g|A_n) \Leftrightarrow f \succeq g.$$

$\succeq_\pi$ is continuous, independent and monotone preference relation, hence, there exists continuous and monotone $I : \mathbb{R}^{|\pi|} \to \mathbb{R}$ that represents $\succeq_\pi$. Define $V(f|\pi) = I(V(f|A_1), \ldots, V(f|A_n))$.

Necessity: Define $f \succeq g$ if and only if $V(f|\pi) \geq V(g|\pi)$. Axiom 1 is straightforward. For Axiom 2, we want to show that any event $A \in \pi$ is aggregating. Thus, if $fAh \succeq gAh$ for some f, g and h in $\mathcal{F}$, then $fAh' \succeq gAh'$ for any $h' \in \mathcal{F}$. Without loss of generality, suppose that we want to demonstrate it for A_1.

$$fA_1h \succeq gA_1h \Leftrightarrow V(fA_1h|\pi) \geq V(gA_1h|\pi)$$
$$\Leftrightarrow I(V(f|A_1), \ldots, V(h|A_n)) \geq I(V(g|A_1), \ldots, V(h|A_n))$$
$$\text{by monotonicity } \Leftrightarrow V(f|A_1) \geq V(g|A_1)$$
$$\text{by monotonicity again} \Leftrightarrow I(V(f|A_1), \ldots, V(h'|A_n)) \geq I(V(g|A_1), \ldots, V(h'|A_n))$$
$$\Leftrightarrow V(fA_1h'|\pi) \geq V(gA_1h'|\pi) \Leftrightarrow fA_1h' \succeq gA_1h'.$$

□

6.3 SASEU

Axiom 2 does not contradict the regular SEU behavior. Thus, if the agent is SEU-maximizer, we will never be able to find any evidence to support or reject the theory. However, if at least one of the functionals $V(\cdot|A_i)$ or $V(\cdot|\pi)$ is non-SEU, then we might observe some interesting features in behavior. Note that if $\phi(\cdot)$ is linear, then the agent is a regular SEU-maximizer.

In order to obtain the SASEU representation, we need to add to the set the classical independence axiom for conditional preferences and for ex-ante preferences over events.

Axiom 3 (Conditional Independence - CI)**.** *For any event* $A \in \pi$*, any acts* $f, g, h \in \mathcal{F}$ *and any* $\alpha \in (0,1)$*:* $f \succeq_A g$ *if and only if* $\alpha f + (1-\alpha)h \succeq_A \alpha g + (1-\alpha)h$*.*

Definition 4. *Suppose that* $\pi = \{A_1, A_2, \ldots, A_n\}$*, then for any act* $f \in \mathcal{F}$ *and event* $A_i \in \pi$ *define acts* $x_f^{A_i}$ *and* f^π*:* $x_f^{A_i} \sim_{A_i} f$ *and* $f^\pi \sim x_f^{A_1} A_1 x_f^{A_2} A_2 \ldots x_f^{A_n}$*.*

Axiom 4 (Ex-ante Independence - EI)**.** *For any acts* $f, g, h \in \mathcal{F}$ *and any* $\alpha \in (0,1)$*:* $f \succeq g$ *if and only if* $\alpha f^\pi + (1-\alpha)h^\pi \succeq \alpha g^\pi + (1-\alpha)h^\pi$*.*

Theorem 4. $\succeq$ *are represented by SASEU with* (π, u, ϕ, P) *and* (π', u', ϕ', P') *if and only if* $u(\cdot)$ *is affine transformation of* $u'(\cdot)$*,* $\phi(\cdot)$ *is affine transformation of* $\phi'(\cdot)$ *and both of them are linear functions.*

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A Appendix

A.1 Proof of Theorem 4

Suppose that there are two different partitions, π and π'. Denote a set $\tilde{\pi} = \{C \in \Sigma : \exists A \in \pi, B \in \pi' : C = A \cap B\}$. Also, denote $\mathcal{A}$ a set of all aggregating events.

Lemma 1. *If $\pi, \pi' \in \mathcal{A}$ and $C \in \tilde{\pi}$, then $C \in \mathcal{A}$.*

Proof. Take events $A \in \pi$ and $B \in \pi'$ such that $C = A \cap B$. Then, $fCx \succeq_A gCx \Leftrightarrow fCx(A \setminus C)h \succeq gCx(A \setminus C)h$. The last relation is equivalent to $fCh(B \setminus C)x(A \setminus C)h \succeq gCh(B \setminus C)x(A \setminus C)h \Leftrightarrow fCh \succeq_B gCh$.

Now, by providing the same argument from event B back to A, one can easily obtain that $fCh \succeq_A gCh$. □

Lemma 2. *Two different SASEU representations of $\succeq$ that satisfy Axioms 1–4 with partitions π and π' exist if there exists a SASEU representation with a denser partition $\tilde{\pi} = \{C : A \in \pi, B \in \pi', A \cap B = C\}$.*

Proof. First, note that $\tilde{\pi} = \{C : A \in \pi, B \in \pi', A \cap B = C\}$ is a partition that consists of aggregating events. Now we just need to show that Axioms 3 and 4 hold for this partition too.

Note that for any $C \in \tilde{\pi}$ we can define conditional preferences $\succeq_C$ because C is an aggregating event. Axiom 3 for $\succeq_C$ follows trivially from Axiom 3 for $\succeq_A$, where $C \subseteq A \in \pi$.

Now we are left to show Ex-ante Independence for $\tilde{\pi}$. Any event $A \in \pi$ consists of a number of events from $\tilde{\pi}$: $A = C_1 \cup C_2 \cup \cdots \cup C_k$. Then for any act $f \in \mathcal{F}$: $f \sim_A x_f^A \sim_A x_f^{C_1} C_1 x_f^{C_2} C_2 \ldots x_f^{C_k}$. By Conditional Independence:

$$\alpha x_f^A + (1-\alpha) x_h^A \sim_A \alpha x_f^{C_1} C_1 x_f^{C_2} C_2 \ldots x_f^{C_k} + (1-\alpha) x_h^{C_1} C_1 x_h^{C_2} C_2 \ldots x_h^{C_k}.$$

Thus, $f \succeq g$ if and only if $\alpha f^\pi + (1-\alpha) h^\pi \succeq g^\pi + (1-\alpha) h^\pi$ if and only if

$$\alpha x_f^{A_1} A_1 x_f^{A_2} A_2 \ldots x_f^{A_n} + (1-\alpha) x_h^{A_1} A_1 x_h^{A_2} A_2 \ldots x_h^{A_n} \succeq \alpha x_g^{A_1} A_1 x_g^{A_2} A_2 \ldots x_g^{A_n} + (1-\alpha) x_h^{A_1} A_1 x_h^{A_2} A_2 \ldots x_h^{A_n}$$

if and only if

$$\alpha x_f^{C_1} C_1 x_f^{C_2} C_2 \ldots x_f^{C_t} + (1-\alpha) x_h^{C_1} C_1 x_h^{C_2} C_2 \ldots x_h^{C_t} \succeq \alpha x_g^{C_1} C_1 x_g^{C_2} C_2 \ldots x_g^{C_t} + (1-\alpha) x_h^{C_1} C_1 x_h^{C_2} C_2 \ldots x_h^{C_t},$$

which is by definition $f^{\tilde{\pi}} \succeq g^{\tilde{\pi}}$. □

Lemma 3. *If $C \in \mathcal{A}$ and $C \subseteq A \in \pi$, then for any act $f \in \mathcal{F}$ and $x_f^C \in X$:*

$$u(x_f^C) = \sum_{s \in C} P(s|C) u(f(s)),$$

where $P(s|C) = \frac{P(s|A)}{\sum_{s \in C} P(s|A)}$.

Proof. Note that $fCh \sim_A x_f^C Ch$ for any $f, h \in \mathcal{F}$, then

$$V_A(fCh) = \phi\left(\sum_{s \in C} P(s|A) u(f(s)) + \sum_{s \in A \setminus C} P(s|A) u(h(s))\right)$$

$$V_A(x_f^C Ch) = \phi\left(\sum_{s \in C} P(s|A) u(x_f^C) + \sum_{s \in A \setminus C} P(s|A) u(h(s))\right).$$

$V_A(fCh) = V_A(x_f^C Ch)$ implies that

$$\sum_{s\in C} P(s|A)u(f(s)) = \sum_{s\in C} P(s|A)u(x_f^C) \Rightarrow$$
$$u(x_f^C) = \sum_{s\in C} \frac{P(s|A)}{\sum_{s\in C} P(s|A)} u(f(s)).$$

□

Proof of Theorem 4. Sufficiency is trivial. Thus, we will show only necessity.

By Lemmas 1 and 2, there exists $\tilde{\pi}$ that also represents $\succeq$. Utility $u(\cdot)$ is defined up to affine transformation and by Lemma 3,

$$u(x_f^C) = \sum_{s\in C} \frac{P(s|A)}{\sum_{s\in C} P(s|A)} u(f(s)).$$

We also know that $\tilde{u}(x_f^C) = \sum_{s\in C} \tilde{P}(s|C)\tilde{u}(f(s))$ and $u(\cdot)$ and $\tilde{u}(\cdot)$ must agree on constant acts. Hence, one is an affine transformation of another. The same argument applies to π'.

Moreover, for $A = C_1 \cup \cdots \cup C_k$ that consists of more than one event from $\tilde{\pi}$,

$$\phi\left(\sum_{s\in A} P(s|A)u(f(s))\right) = \sum_{C_i\in A} \tilde{P}(C_i|A)\tilde{\phi}\left(\sum_{s\in C_i} \frac{P(s|A)}{\sum_{s\in C_i} P(s|A)} u(f(s))\right).$$

The latter implies that $\phi(\cdot)$ and $\tilde{\phi}(\cdot)$ are linear and affine transformations of each other.

□

A.2 Proof of Theorem 1

The agent purchases a bundle of Arrow securities that maximizes her value given a certain amount of income I and the price p_i of an Arrow security that pays 1 in state i:

$$V(x|\pi) \to \max_x$$
$$\text{s.t. } \sum_i p_i x_i = I.$$

Hence, if λ is a Lagrange multiplier, then the first order condition for each state $s \in A$ is

$$p_A \phi' \left(\sum_{s\in A} P(s|A) u(x_s) \right) p(s|A) u'(x_s) = \lambda p_s^x.$$

Pick two states s_i and s_j.

Two states from the same event: If s_i and s_j belong to the same event A, then

$$\frac{p(s_i|A)}{p(s_j|A)} \frac{u'(x_{s_i})}{u'(x_{s_j})} = \frac{p_{s_i}^x}{p_{s_j}^x}. \tag{9}$$

Now choose two other bundles y and z such that $y_{s_i} = x_{s_i}$, $z_{s_j} = x_{s_j}$, but $y_{s_j} \neq x_{s_j}$ and $z_{s_i} \neq x_{s_i}$. Then we obtain

$$\frac{u'(y_{s_j})}{u'(x_{s_j})} = \frac{p_{s_i}^x}{p_{s_j}^x} \frac{p_{s_j}^y}{p_{s_i}^y} \text{ and } \frac{u'(x_{s_i})}{u'(z_{s_i})} = \frac{p_{s_i}^x}{p_{s_j}^x} \frac{p_{s_j}^z}{p_{s_i}^z}. \tag{10}$$

Note that both left-side ratios depend only on payoffs at a related state and nothing else. Thus, if payoffs at other states are changed, it must not affect the above ratios.

Two states from different events: We repeat the above derivation when states s_i and s_j are from different events A_i and A_j:

$$\frac{p_{A_i} P(s_i|A_i)}{p_{A_j} p(s_j|A_j)} \frac{\phi' \left(\sum_{s\in A_i} P(s|A_i) u(x_s) \right) u'(x_{s_i})}{\phi' \left(\sum_{s\in A_j} P(s|A_j) u(x_s) \right) u'(x_{s_j})} = \frac{p_{s_i}^x}{p_{s_j}^x}. \tag{11}$$

After taking bundles y and z as before, we get

$$\frac{\phi' \left(\sum_{s\in A_j} P(s|A_j) u(y_s) \right) \phi' \left(\sum_{s\in A_i} P(s|A_i) u(x_s) \right)}{\phi' \left(\sum_{s\in A_i} P(s|A_i) u(y_s) \right) \phi' \left(\sum_{s\in A_j} P(s|A_j) u(x_s) \right)} \frac{u'(y_{s_j})}{u'(x_{s_j})} = \frac{p_{s_i}^x}{p_{s_j}^x} \frac{p_{s_j}^y}{p_{s_i}^y} \tag{12}$$

$$\frac{\phi' \left(\sum_{s\in A_j} P(s|A_j) u(z_s) \right) \phi' \left(\sum_{s\in A_i} P(s|A_i) u(x_s) \right)}{\phi' \left(\sum_{s\in A_i} P(s|A_i) u(z_s) \right) \phi' \left(\sum_{s\in A_j} P(s|A_j) u(x_s) \right)} \frac{u'(x_{s_i})}{u'(z_{s_i})} = \frac{p_{s_i}^x}{p_{s_j}^x} \frac{p_{s_j}^z}{p_{s_i}^z}. \tag{13}$$

Notice that the left sides of BOTH above equalities do not depend on other states if and only if both s_i and s_j are singleton events. Thus, potentially, we might get confused and aggregate all singletons together. However, consider FOC between a singleton s_i and a state s_j from non-singleton event A_j:

$$\frac{\phi'\left(\sum_{s\in A_j} P(s|A_j)u(y_s)\right)\phi'\left(u(x_{s_i})\right)}{\phi'\left(u(y_{s_i})\right)\phi'\left(\sum_{s\in A_j} P(s|A_j)u(x_s)\right)}\frac{u'(y_{s_j})}{u'(x_{s_j})}=\frac{p^x_{s_i}}{p^x_{s_j}}\frac{p^y_{s_j}}{p^y_{s_i}}. \tag{14}$$

Note that the left side depends on payoffs in all states at A_j and s_i, however, it does not depend on the payoffs at other singletons. Hence, after recognizing the groups of the states that might be potential events, we are able to identify which group is the group of singletons. Thus, the partition π is identified.

Now note that by choosing different values of x_{s_i}, x_{s_j}, y_{s_j} and z_{s_i} when s_i and s_j belong to the same event, we can identify $u(\cdot)$ up to affine transformation. After that we consider the original first order condition again and identify the probability ratios $\frac{P(s_i|A)}{P(s_j|A)}$. Given that $\sum_{s|A} P(s|A) = 1$, we can identify the probabilities.

In order to identify $\phi(\cdot)$, we consider states from different events. $u(\cdot)$ and all conditional probabilities have already been identified, hence, we can obtain the values of different ratios of the kind

$$\frac{\phi'\left(\sum_{s\in A_j} P(s|A_j)u(y_s)\right)\phi'\left(\sum_{s\in A_i} P(s|A_i)u(x_s)\right)}{\phi'\left(\sum_{s\in A_i} P(s|A_i)u(y_s)\right)\phi'\left(\sum_{s\in A_j} P(s|A_j)u(x_s)\right)}.$$

Note that if A_i is a singleton and y_{s_i} was chosen such that $y_{s_i} = x_{s_i}$, then $\phi'\left(\sum_{s\in A_i} P(s|A_i)u(x_s)\right)$ and $\phi'\left(\sum_{s\in A_i} P(s|A_i)u(y_s)\right)$ cancel each other out. If A_i is not a singleton, $\phi'\left(\sum_{s\in A_i} P(s|A_i)u(x_s)\right)$ and $\phi'\left(\sum_{s\in A_i} P(s|A_i)u(y_s)\right)$ can be cancel out by choosing $y_s = x_s$ for all $s \in A_i$. Hence, we can identify $\phi(\cdot)$ up to affine transformation.

The only unknown variables left are priors about events p_{A_i}. However, we can obtain $\frac{p_{A_i}}{p_{A_j}}$ from the first order condition for two states from different events (????). Finally, given that $\sum_i p_{A_i} = 1$, we identify p_{A_i} as well.

A.3 Proof of Theorem 2

Lemma 4. *Suppose that $\pi = \{A, s\}$, where $A = \{s_1, s_2, \ldots, s_k\}$, and $\tilde{\pi} = \{A\backslash s_1, s_1, s\}$. Assume also that both $u(\cdot)$ and $\phi(\cdot)$ are differentiable concave functions. If c_s denotes consumption in* *state s under π, while $\tilde{c}_s$ is consumption under $\tilde{\pi}$, then one of the following holds:*

1. $\tilde{c}_s > c_s$, $\tilde{c}_1 < c_1$, *and* $\tilde{c}_i > c_i$, *where* $i \neq 1$;
2. $\tilde{c}_s > c_s$, $\tilde{c}_1 > c_1$, *and* $\tilde{c}_i < c_i$, *where* $i \neq 1$;
3. $\tilde{c}_s < c_s$, $\tilde{c}_1 < c_1$, *and* $\tilde{c}_i > c_i$, *where* $i \neq 1$;
4. $\tilde{c}_s < c_s$, $\tilde{c}_1 > c_1$, *and* $\tilde{c}_i < c_i$, *where* $i \neq 1$.

Proof. Denote $V_A(c) = \sum_{s_i \in A} P(s_i|A)u(c_i)$, then the first order conditions between s_1 and the other states $s_i \in A$ for the SASEU agent under π can be rewritten as

$$\frac{p_1}{p_i}\frac{P(s_i)}{P(s_1)} = \frac{u'(c_1)}{u'(c_i)}. \tag{15}$$

While the first order conditions between s_1 and the other states in A for the SASEU agent under $\tilde{\pi}$ are

$$\frac{p_1}{p_i}\frac{P(s_i)}{P(s_1)} = \frac{\phi'(u(\tilde{c}_1))}{\phi'(V_{A\setminus s_1}(\tilde{c}))}\frac{u'(\tilde{c}_1)}{u'(\tilde{c}_i)}. \tag{16}$$

The first order condition between outside state s and the states in A (including s_1) under π is

$$\frac{p_i}{p_s}\frac{P(s)}{P(s_i)} = \frac{\phi'(V_A(c))}{\phi'(u(c_s))}\frac{u'(c_i)}{u'(c_s)}. \tag{17}$$

The first order condition between outside state s and s_1 under $\tilde{\pi}$ is

$$\frac{p_1}{p_s}\frac{P(s)}{P(s_1)} = \frac{\phi'(u(\tilde{c}_1))}{\phi'(u(\tilde{c}_s))}\frac{u'(\tilde{c}_1)}{u'(\tilde{c}_s)}. \tag{18}$$

The first order condition between outside state s and the other states in A under $\tilde{\pi}$ are

$$\frac{p_i}{p_s}\frac{P(s)}{P(s_i)} = \frac{\phi'(V_{A\setminus s_1}(\tilde{c}))}{\phi'(u(\tilde{c}_s))}\frac{u'(\tilde{c}_i)}{u'(\tilde{c}_s)}. \tag{19}$$

The first order condition between states in $A\backslash s_1$ under π and $\tilde{\pi}$ are

$$\frac{p_i}{p_j}\frac{P(s_j)}{P(s_i)} = \frac{u'(c_i)}{u'(c_j)} \text{ and } \frac{p_i}{p_j}\frac{P(s_j)}{P(s_i)} = \frac{u'(\tilde{c}_i)}{u'(\tilde{c}_j)} \tag{20}$$

$$\Rightarrow \frac{u'(c_i)}{u'(c_j)} = \frac{u'(\tilde{c}_i)}{u'(\tilde{c}_j)}. \tag{21}$$

The above conditions imply the following:

$$\frac{\phi'(u(\tilde{c}_s))}{\phi'(u(c_s))}\frac{u'(\tilde{c}_s)}{u'(c_s)} = \frac{\phi'(V_{A\backslash s_1}(\tilde{c}))}{\phi'(V_A(c))}\frac{u'(\tilde{c}_i)}{u'(c_i)} = \frac{\phi'(u(\tilde{c}_1))}{\phi'(V_A(c))}\frac{u'(\tilde{c}_1)}{u'(c_1)}. \tag{22}$$

Now we consider different cases.

1. $\frac{\phi'(u(\tilde{c}_s))}{\phi'(u(c_s))}\frac{u'(\tilde{c}_s)}{u'(c_s)} < 1$ and $\frac{u'(\tilde{c}_1)}{u'(c_1)} > \frac{u'(\tilde{c}_i)}{u'(c_i)}$.

   Concave $u(\cdot)$ and $\phi(\cdot)$ imply that $\tilde{c}_s > c_s$. Now suppose that $\tilde{c}_1 \geq c_1$, then $1 \geq \frac{u'(\tilde{c}_1)}{u'(c_1)} > \frac{u'(\tilde{c}_i)}{u'(c_i)}$ implying that $\tilde{c}_1 > c_1$ and $\tilde{c}_i > c_i$. However, note that the FOC for the states in $A\backslash s_1$ guarantees that if $\tilde{c}_i > c_i$, then $\tilde{c}_j > c_j$ for all $i, j \in A\backslash s_1$. Which leads us to contradiction, because consumption in all states cannot go up without the change in prices or income. Hence, $\tilde{c}_1 < c_1$.

   Now we are left to show that $\tilde{c}_i > c_i$. First, note that $\frac{u'(\tilde{c}_1)}{u'(c_1)} > \frac{u'(\tilde{c}_i)}{u'(c_i)}$ implies $\phi'(V_{A\backslash s_1}(\tilde{c})) > \phi'(u(\tilde{c}_1))$, so $u(c_1) > u(\tilde{c}_1) > V_{A\backslash s_1}(\tilde{c})$. Then two situations are possible: (1) If $V_{A\backslash s_1}(\tilde{c}) > V_A(c)$, then

$$V_A(c) = P(s_1|A)u(c_1) + (1 - P(s_1|A))V_{A\backslash s_1}(c) > P(s_1|A)V_{A\backslash s_1}(\tilde{c}) + (1 - P(s_1|A))V_{A\backslash s_1}(c)$$
$$\Rightarrow V_{A\backslash s_1}(\tilde{c}) > V_{A\backslash s_1}(c).$$

   According to the FOC for the states in $A\backslash s_1$, consumption in all states moves in the same direction, hence, implying that $\tilde{c}_i > c_i$. (2) If $V_{A\backslash s_1}(\tilde{c}) < V_A(c)$, then $\frac{u'(\tilde{c}_i)}{u'(c_i)} < 1$ implying that $\tilde{c}_i > c_i$ anyway.

   Thus, we obtain the case 1 in the lemma, where $\tilde{c}_s > c_s$, $\tilde{c}_1 < c_1$, and $\tilde{c}_i > c_i$, where $i \neq 1$.

2. $\frac{\phi'(u(\tilde{c}_s))}{\phi'(u(c_s))}\frac{u'(\tilde{c}_s)}{u'(c_s)} < 1$ and $\frac{u'(\tilde{c}_1)}{u'(c_1)} < \frac{u'(\tilde{c}_i)}{u'(c_i)}$.

Concave $u(\cdot)$ and $\phi(\cdot)$ imply that $\tilde{c}_s > c_s$. Now suppose that $\tilde{c}_i \geq c_i$, then $\frac{u'(\tilde{c}_1)}{u'(c_1)} < \frac{u'(\tilde{c}_i)}{u'(c_i)} \leq 1$ implying that $\tilde{c}_1 > c_1$ and $\tilde{c}_i > c_i$. However, note that the FOC for the states in $A \backslash s_1$ guarantees that if $\tilde{c}_i > c_i$, then $\tilde{c}_j > c_j$ for all $i, j \in A \backslash s_1$. Which leads us to contradiction, because consumption in all states cannot go up without the change in prices or income. Hence, $\tilde{c}_i < c_i$ for all $s_i \in A \backslash s_1$.

Now we are left to show that $\tilde{c}_1 > c_1$. First, note that $\frac{u'(\tilde{c}_i)}{u'(c_i)} > 1$ implies $\phi'(V_{A\backslash s_1}(\tilde{c})) < \phi'(V_A(c))$, so $V_{A\backslash s_1}(c) > V_{A\backslash s_1}(\tilde{c}) > V_A(c) \Rightarrow u(c_1) < V_{A\backslash s_1}(c)$. Then two situations are possible: (1) If $u(\tilde{c}_1) > V_A(c)$, then

$$V_A(c) = P(s_1|A)u(c_1) + (1 - P(s_1|A))V_{A\backslash s_1}(c) > u(c_1)$$
$$\Rightarrow u(\tilde{c}_1) > u(c_1).$$

(2) If $u(\tilde{c}_1) < V_A(c)$, then $\frac{u'(\tilde{c}_1)}{u'(c_1)} < 1$ implying that $\tilde{c}_1 > c_1$ anyway.

Thus, we obtain the case 2 in the lemma, where $\tilde{c}_s > c_s$, $\tilde{c}_1 > c_1$, and $\tilde{c}_i < c_i$, where $i \neq 1$.

3. $\frac{\phi'(u(\tilde{c}_s))}{\phi'(u(c_s))}\frac{u'(\tilde{c}_s)}{u'(c_s)} > 1$ and $\frac{u'(\tilde{c}_1)}{u'(c_1)} > \frac{u'(\tilde{c}_i)}{u'(c_i)}$.

   Concave $u(\cdot)$ and $\phi(\cdot)$ imply that $\tilde{c}_s < c_s$. Now suppose that $\tilde{c}_i \leq c_i$, then $\frac{u'(\tilde{c}_1)}{u'(c_1)} > \frac{u'(\tilde{c}_i)}{u'(c_i)} \geq 1$ implying that $\tilde{c}_1 < c_1$ and $\tilde{c}_i \leq c_i$. However, note that the FOC for the states in $A \backslash s_1$ guarantees that if $\tilde{c}_i < c_i$, then $\tilde{c}_j < c_j$ for all $i, j \in A \backslash s_1$. Which leads us to contradiction, because consumption in all states cannot go down without the change in prices or income. Hence, $\tilde{c}_i > c_i$ for all $s_i \in A \backslash s_1$.

   Now we are left to show that $\tilde{c}_1 < c_1$. First, note that $\frac{u'(\tilde{c}_i)}{u'(c_i)} < 1$ implies $\phi'(V_{A\backslash s_1}(\tilde{c})) > \phi'(V_A(c))$, so $V_{A\backslash s_1}(c) < V_{A\backslash s_1}(\tilde{c}) < V_A(c) \Rightarrow u(c_1) > V_{A\backslash s_1}(c) \Rightarrow u(c_1) > V_A(c)$. Then two situations are possible: (1) If $u(\tilde{c}_1) < V_A(c)$, and $u(c_1) > V_A(c) \Rightarrow u(c_1) > u(\tilde{c}_1)$; (2) If $u(\tilde{c}_1) > V_A(c)$, then $\frac{u'(\tilde{c}_1)}{u'(c_1)} > 1$ implying that $\tilde{c}_1 < c_1$ anyway.

   Thus, we obtain the case 3 in the lemma, where $\tilde{c}_s < c_s$, $\tilde{c}_1 < c_1$, and $\tilde{c}_i > c_i$, where $i \neq 1$.

4. $\frac{\phi'(u(\tilde{c}_s))}{\phi'(u(c_s))}\frac{u'(\tilde{c}_s)}{u'(c_s)} > 1$ and $\frac{u'(\tilde{c}_1)}{u'(c_1)} < \frac{u'(\tilde{c}_i)}{u'(c_i)}$.

   Concave $u(\cdot)$ and $\phi(\cdot)$ imply that $\tilde{c}_s < c_s$. Now suppose that $\tilde{c}_1 \leq c_1$, then $1 \leq \frac{u'(\tilde{c}_1)}{u'(c_1)} < \frac{u'(\tilde{c}_i)}{u'(c_i)}$ implying that $\tilde{c}_1 \leq c_1$ and $\tilde{c}_i < c_i$. However, note that the FOC for the states in $A \backslash s_1$

guarantees that if $\tilde{c}_i < c_i$, then $\tilde{c}_j < c_j$ for all $i, j \in A \backslash s_1$. Which leads us to contradiction, because consumption in all states cannot go down without the change in prices or income. Hence, $\tilde{c}_1 > c_1$.

Now we are left to show that $\tilde{c}_i < c_i$. First, note that $\frac{u'(\tilde{c}_1)}{u'(c_1)} < \frac{u'(\tilde{c}_i)}{u'(c_i)}$ implies $\phi'(V_{A\backslash s_1}(\tilde{c})) < \phi'(u(\tilde{c}_1))$, so $u(c_1) < u(\tilde{c}_1) < V_{A\backslash s_1}(\tilde{c})$. Then two situations are possible: (1) If $V_{A\backslash s_1}(\tilde{c}) < V_A(c)$, then

$$V_A(c) = P(s_1|A)u(c_1) + (1 - P(s_1|A))V_{A\backslash s_1}(c) < P(s_1|A)V_{A\backslash s_1}(\tilde{c}) + (1 - P(s_1|A))V_{A\backslash s_1}(c)$$
$$\Rightarrow V_{A\backslash s_1}(\tilde{c}) < V_{A\backslash s_1}(c).$$

According to the FOC for the states in $A\backslash s_1$, consumption in all states moves in the same direction, hence, implying that $\tilde{c}_i < c_i$. (2) If $V_{A\backslash s_1}(\tilde{c}) > V_A(c)$, then $\frac{u'(\tilde{c}_i)}{u'(c_i)} > 1$ implying that $\tilde{c}_i < c_i$ anyway.

Thus, we obtain the case 4 in the lemma, where $\tilde{c}_s < c_s$, $\tilde{c}_1 > c_1$, and $\tilde{c}_i < c_i$, where $i \neq 1$.

□

Lemma 5. *Suppose that $\pi = \{A_1, A_2, \ldots, A_k\}$ and $\tilde{\pi} = \{A_1\backslash s_1, s_1, A_2, \ldots, A_k\}$. Assume also concave and differentiable $u(\cdot)$ and $\phi(\cdot)$. Then for any event A_i such that $i \neq 1$ one of the following holds:*

1. *$\tilde{c}_s > c_s$ for all $s \in A_i$;*
2. *$\tilde{c}_s < c_s$ for all $s \in A_i$.*

Proof. Consider the first order condition between the states s_i and s_j inside event A_i, note that it is not affected by disaggregation of s_1:

$$\frac{u'(c_j)}{u'(c_i)} = \frac{p_j}{p_i}\frac{P(s_i)}{P(s_j)} = \frac{u'(\tilde{c}_j)}{u'(\tilde{c}_i)}.$$

Thus, if $\tilde{c}_j > c_j$, then $\tilde{c}_i > c_i$, and vice versa. Given that the condition is the same for all states in A_i, we obtain that consumption in all states inside A_i moves in the same direction. □

Lemma 6. *Suppose that* $\pi = \{A_1, A_2, \ldots, A_k\}$ *and* $\tilde{\pi} = \{A_1 \backslash s_1, s_1, A_2, \ldots, A_k\}$*. Assume also concave and differentiable* $u(\cdot)$ *and* $\phi(\cdot)$*. Then for all* $s \in A_i$ *and all* A_i *such that* $i \neq 1$ *only one of the following holds:*

1. $\tilde{c}_s > c_s$*;*

2. $\tilde{c}_s < c_s$*.*

Proof. Note that the first order conditions between the states from the unaffected events stay the same:

$$\frac{\phi'(V_{A_i}(c))}{\phi'(V_{A_j}(c))}\frac{u'(c_i)}{u'(c_j)} = \frac{p_i}{p_j}\frac{P(s_j)}{P(s_i)} = \frac{\phi'(V_{A_i}(\tilde{c}))}{\phi'(V_{A_j}(\tilde{c}))}\frac{u'(\tilde{c}_i)}{u'(\tilde{c}_j)},$$

where $s_j \in A_j$ and $s_i \in A_i$, $j, i \neq 1$. Also by previous lemma, we know that if $\tilde{c}_i > c_i$, then $V_{A_i}(\tilde{c}) > V_{A_i}(c)$, and vice versa. The same holds for consumption in the state s_j. The result follows. □

Lemma 7. *Consider two partitions* π *and* $\tilde{\pi}$ *such that* $\tilde{\pi} = \{A_1 \backslash s_1, s_1, A_2, \ldots, A_k\}$ *and* $\pi = \{A_1, A_2, \ldots, A_k\}$*. In addition, suppose that both* $u(\cdot)$ *and* $\phi(\cdot)$ *are concave and differentiable. Then*

1. $u(c_1) > V_{A \backslash s_1}(c) \Leftrightarrow u(\tilde{c}_1) > V_{A \backslash s_1}(\tilde{c}) \Leftrightarrow \tilde{c}_1 < c_1, \tilde{c}_i > c_i$ *for any* $s_i \in A \backslash s_1$*;*

2. $u(c_1) > V_{A \backslash s_1}(c) \Leftrightarrow u(\tilde{c}_1) > V_{A \backslash s_1}(\tilde{c}) \Leftrightarrow \tilde{c}_1 < c_1, \tilde{c}_i > c_i$ *for any* $s_i \in A \backslash s_1$*.*

Moreover, one of the following holds:

1. $\tilde{c}_s > c_s$ *for all* $s \in A_k$ *and all* $k \neq 1$*;*

2. $\tilde{c}_s < c_s$ *for all* $s \in A_k$ *and all* $k \neq 1$*.*

Proof. The last lemma implies that consumption in unaffected events move in the same direction, then together with Lemma 4, the proof of this lemma follows. □

The proof of the theorem follows from the fact that consumption in the states in B moves in the same direction, because FOC between these states is not affected. It implies that $V_B(c)$ moves together with consumption in any of the states in B. By using this result and previous lemma, the rest follows.