Economics Working Paper Series

2017 - 03

Nominal GDP Targeting with Heterogeneous Labor Supply

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February 2017

Nominal GDP Targeting with Heterogeneous Labor Supply

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This version: 13 February 2017‡

Abstract

We study nominal GDP targeting as optimal monetary policy in a model with a credit market friction following Azariadis, Bullard, Singh and Suda (2016), henceforth ABSS. As in ABSS, the macroeconomy we study has considerable income inequality which gives rise to a large private sector credit market. Households participating in this market use non-state contingent nominal contracts (NSCNC). We extend the ABSS framework to allow for endogenous and heterogeneous household labor supply among credit market participant households. We show that nominal GDP targeting continues to characterize optimal monetary policy in this setting. Optimal monetary policy repairs the distortion caused by the credit market friction and so leaves heterogeneous households supplying their desired amount of labor, a type of "divine coincidence" result. We also analyze the case when there is an aging population. We interpret these findings in light of the recent debate in monetary policy concerning labor force participation.

Keywords: Non-state contingent nominal contracting, optimal monetary policy, nominal GDP targeting, life cycle economies, heterogeneous households, credit market participation, labor supply. *JEL codes*: E4, E5.

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‡We thank Thomas Sargent, Chung-Shu Wu, Been-Lon Chen and seminar participants at Bank of Korea's 2016 Policy Conference, Academia Sinica, Society for Computational Economics and Finance 2016 and London School of Economics for their constructive input.

1 Nominal GDP targeting

1.1 Overview

Recent papers by Sheedy (2014), Koenig (2013), and Azariadis, Bullard, Singh and Suda (2016), hereafter ABSS, provide analyses of optimal monetary policy in economies where the key friction is in the credit market in the form of non-state contingent nominal contracting (NSCNC). They all show that optimal policy can be characterized as a version of nominal GDP targeting in that environment. The monetary policy provides a form of insurance to private sector credit-using households.

The ABSS model is based on credit-using households with inelastic labor supply. An open question is whether the optimal monetary policy they isolate could continue to be characterized as nominal GDP targeting if credit-using households were allowed to adjust labor supply in response to shocks. In principle, these (heterogeneous) households may be able to partially self-insure in this circumstance, thus altering the nature of the nominal GDP targeting policy or even rendering it unnecessary. Our goal is to study this issue in this paper.

We construct an extension of the life cycle framework of ABSS (2016) to a case of endogenous (and heterogeneous) labor supply. Credit-using households have homothetic preferences defined over consumption and leisure choices. We also include population growth in the model, in order to be able to comment on whether the changing nature of the workforce would have any implications for our findings. We compare our findings to some labor market data for the U.S. economy, but we do not consider this version of the model to be sophisticated enough to compare to data in a more comprehensive way.[1]

Our main finding is that nominal GDP targeting continues to characterize the optimal monetary policy in the situation with endogenous and heterogeneous labor supply and constant population growth. The policy completely repairs the distortion caused by the NSCNC friction and allows all credit-using households to consume equal amounts at each date. This is the hallmark of the NGDP targeting policy in this model—under this policy credit markets are

[1] We hope to take up the challenge of matching data more comprehensively in future versions of the model.

characterized by "equity share" contracting, which is optimal when preferences are homothetic.

Our main result is a version of the "divine coincidence" result familiar from the New Keynesian monetary policy literature.[2] In our model, there is a single friction, which is non-state contingent nominal contracting in the credit sector. Monetary policy can alter the price level to eliminate the distortion arising from this friction and restore the first-best allocation of resources. Our main result shows that, in this situation, households are able to choose their optimal level of labor supply as well, and in fact these heterogeneous labor supply choices are independent of the aggregate shock in the model. The divine coincidence is that, by completely mitigating the credit market friction, the monetary policy also allows for optimal labor supply choices.

Our main result suggests that optimal monetary policy can be conducted in this environment without reference to labor market outcomes. Labor supply growth would in this situation be closely related to labor force growth, as it is in the U.S. data. An economy in this class could have an aging workforce if the labor force growth rate is slowing over time. In this situation the comparative statics of the model predict that as the workforce ages, younger cohorts will work less and older cohorts will work more. This is indeed what has happened in the U.S. data over the last 20 years, as we show near the end of the paper. We take this as one indication that the forces influencing labor supply in this model may also be influencing labor supply in the U.S. economy.

1.2 Additional motivation

Aside from the issue of whether the NGDP targeting policy remains optimal in the face of endogenous labor supply in this setting, we additionally motivate the paper with a contemporary issue in monetary policy. Since the 2007-2009 financial crisis, the labor force participation rate in the U.S. has been low and falling.[3] A key question for monetary policymakers has been whether the falling labor force participation rate is driven by business cycle factors, in which case

[2] See Blanchard and Gali (2010) and Woodford (2003).

[3] The literature documenting the fall in the labor force participation rate is growing. See for example, Aaronson, Cajner, Fallick, Galbis-Reig, Smith and Wascher (2016), Van Zandweghe (2012), Daly, Elias, Hobijn and Jordà (2012), Hotchkiss and Rios-Avila (2013) among others.

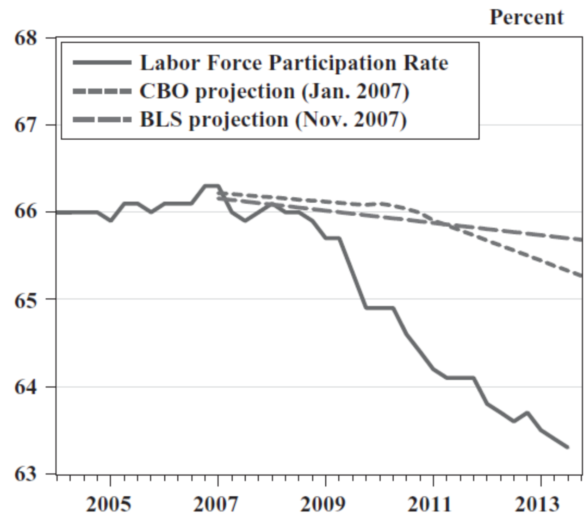

Figure 1: The first figure, left panel, from Erceg and Levin (2014). After the financial crisis and recession of 2007-2009, labor force participation in the U.S. fell more than its forecast by leading government agencies.

monetary policymakers may want to attempt to increase the participation rate through monetary policy choices. But an alternative, and we think more traditional, view is that the labor force participation rate is driven by demographic factors, in which case policymakers will be unable to meaningfully change the participation rate via monetary policy. The results in this paper provide some support for the traditional view.

Figure 1 is the first figure from Erceg and Levin (2014). They suggest, based on this evidence from 2004-2013, that labor force participation fell much more than expected by some government agencies in the aftermath of the financial crisis and recession of 2007-2009 in the U.S. They construct a New Keynesian model of monetary policy in which the labor force participation rate would not be an important cyclical variable in normal times, but which may remain significantly depressed following a particularly large macroeconomic shock. They find that monetary policy may be able to help mitigate an inefficiently low level of labor force participation.

Figure 2 offers a different take on the same data. This Figure again shows the U.S. labor force participation rate, the solid line, this time from 1991-2015.

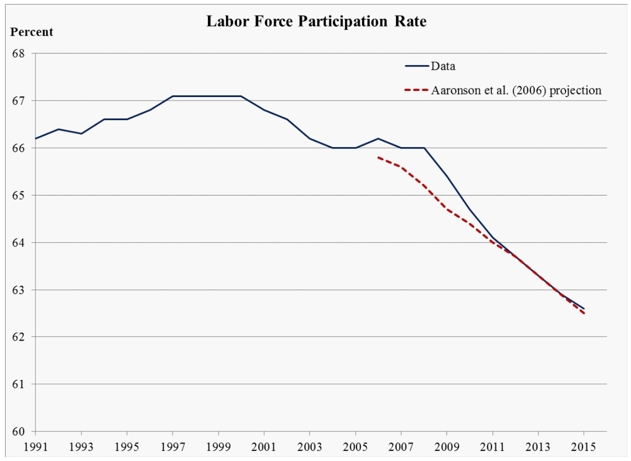

Figure 2: Labor force participation in the U.S., 1991-2015. The dotted line represents the forecast made by Aaronson et al (2006) using a model based largely on demographic factors. The forecast is essentially correct even 7 to 9 years in the future.

The figure also shows a forecast labor force participation rate due to Aaronson, Fallick, Figura, Pingle, and Wascher (2006), represented by the dotted line in the figure. The Aaronson, *et al.*, (2006) model is based importantly on demographic effects. Their 2006 model successfully predicted the decline in labor force participation as of the 2013 to 2015 time frame, about 7 to 9 years in the future.[4] We think of the Aaronson, *et al.*, (2006) findings as consistent with what we see as a "traditional view" in macroeconomics, whereby labor force participation is an essentially acyclical variable, movements in which would not be indicative of business cycle developments but would rather reflect longer-run changes in desired labor supply by the workforce in the economy. These longer-run movements would then be viewed as largely independent of monetary policy choices.

In the current paper, we will present a model which is consistent with what we call the traditional view of labor force participation. In the model, monetary policy will have an important role to play in ensuring good credit market perfor-

[4] Hall and Petrosky-Nadeau (2016), Daly et. al. (2012) among others also attribute declining labor force participation rates to ongoing secular change in trend. In a recent paper, Krueger (2016) plots forecast of labor force participation rate from 2007 to 2016 in Figure 3 and the plot exhibits a similar declining pattern.

mance. If this monetary policy is carried out in the optimal manner (which will turn out to be our version of nominal GDP targeting), then labor supply choices, while heterogeneous, will indeed be independent of monetary policy and of the shocks buffeting the economy.[5] Furthermore, we will have population growth in the model, and we will use this feature to show how changing demographics may cause changing labor supply phenomena in the economy while not interfering with the central bank's ability to deliver a first-best policy.[6]

2 Environment

2.1 The private sector

2.1.1 Background on symmetry

ABSS (2016) work with a stylized general equilibrium life cycle economy with an aggregate shock to productivity growth. This means that the economy has heterogeneous households along with an aggregate shock, and therefore that the equilibrium includes tracking the asset-holding distribution in the economy. However, to keep the heterogeneity manageable and the equilibrium calculable in a tractable way, ABSS (2016) made certain "stylized symmetry assumptions." In the life cycle model, much depends on the productive capacity of older cohorts, the suppliers of credit, versus the productive capacity of younger cohorts, the demanders of credit. The goal is to keep this aspect of the model in balance via simplifying assumptions. Accordingly, the productivity endowments of credit-using households are assumed to be perfectly symmetric, peaking in the middle period of life. In addition, there is no discounting of the future in the preferences (that is, the discount factor $\beta = 1$), which would otherwise tend to favor consumption today over consumption tomorrow.[7] Combined with time-separable log preferences, these assumptions help make the key quantities in the model—including the net asset positions of all households—linear in the

[5] Here the phrase "independent of monetary policy" is meant to convey the idea that, while the policymaker is changing the price level in response to aggregate productivity shocks, the households do not wish to change their labor supply in response to the price level movements.

[6] For an extensive discussion of demographic trends and their associated effects on labor force participation, see Krueger (2016). For a review of some of the recent literature and connections to current monetary policy issues, see Bullard (2014).

[7] We could add discounting back in, at the cost of minor complications. It is unnecessary in this type of model and so we omit it for simplicity.

real wage. This means that the asset-holding distribution can be tracked easily. The equilibrium real interest rate on loans is then exactly equal to the real output growth rate each period, even in the stochastic case. Altogether, this creates a particularly tractable framework in which the intuition behind the nominal GDP targeting policy is brought into clearer relief.

We will maintain these same stylized symmetry assumptions in this paper as well. However, by adding population growth in this paper, the symmetry underlying the ABSS (2016) model will be slightly disturbed. In particular, with constant population growth there will be "too much" productive capacity in the younger cohorts (that is, too much population mass in those cohorts given the productivity endowments), and not enough in the older cohorts. To restore symmetry, we will therefore allow publicly-issued debt as an asset in this paper in order to again create an equilibrium where the real interest rate on loans is equal to the real output growth rate each period, as in ABSS (2016). The fiscal authority will issue just enough debt each period to pay principal and interest on previously-issued debt, and this assumption will have no other role to play in this version of the model.

2.1.2 Cohorts and segmented markets

We now turn to describing the model in more detail. Following ABSS (2016), we use a life cycle model with segmented markets. Cohorts are collections of identical, atomistic households entering and later exiting the economy at the same date. Cohorts are divided into two types, a large group of "credit market participants" and a small group of "credit market non-participants." We also refer to these two groups as "credit users" and "cash users," respectively. Households live in discrete time for $T+1$ periods. Our results will hold for any integer $T \geq 2$, but for most presentation purposes we will use the value $T+1 = 241$ so that we can interpret results as corresponding to a quarterly model in which households begin and end economic life with zero assets, starting, let's say, around age 20, and continuing until death. Choosing $T+1$ to be an odd number allows for a convenient and specific peak period for participant productivity endowment profiles. The economy itself continues into the infinite past and into the infinite future, with discrete time denoted by t where $-\infty < t < \infty$. The assets in the economy are privately-issued nominal debt, publicly-issued nominal debt, and currency. Credit market participant households can hold any of these assets, but in the equilibria we study, they will only hold the two types of debt, which will each pay an identical real rate of return always higher than the real rate of return on holding currency.[8] Cash-using households are excluded from credit markets altogether and only hold currency. All loan contracts are for one period, are not state-contingent, and are expressed in nominal terms—we call this the non-state contingent nominal contracting (NSCNC) friction.

The extension in this paper relative to ABSS (2016) is to add endogenous labor supply and population growth.

2.1.3 Population growth

We think of the participant portion of the cohort entering the economy at date t as having measure n_t, for any date t.[9] These cohort sizes are related by a constant gross growth rate $\psi \geq 1$, as

$$n_t = \psi n_{t-1} \tag{1}$$

with $n_0 > 0$. The participant population at date t can then be measured as

$$N(t) = n_t + n_{t-1} + \cdots + n_{t-T} \tag{2}$$

$$= n_t \left[1 + \psi^{-1} + \psi^{-2} + \cdots + \psi^{-T}\right]. \tag{3}$$

Similarly the participant population at $t-1$ would be

$$N(t-1) = n_{t-1} \left[1 + \psi^{-1} + \psi^{-2} + \cdots + \psi^{-T}\right]. \tag{4}$$

The gross participant population growth rate is therefore equal to $N(t) /\!\!/ N(t-1) = \psi$.

We think of the relatively small non-participant portion of the cohort entering the economy at date t symmetrically. They have measure m_t at date t, a

[8] That is, nominal interest rates will always be positive.

[9] We generally follow the notational convention that subscript t indicates the cohort (the "birth date") and that t in parentheses denotes real time. The exception to this is the productivity endowment notation as described below.

constant gross growth rate of ψ, and a gross non-participant population growth rate of ψ. Since $m_t + n_t = 1$, the overall population growth rate is also ψ. Since all households work at least some of the time in this model, the population growth rate is also the labor force growth rate.

2.1.4 Participant productivity endowments

Credit market participant households enter the economy endowed with a known sequence of productivity units given by $e = \{e_s\}_{s=0}^{T}$. This notation means that each household entering the economy has productivity endowment e_0 in the first period of activity, e_1 in the second, and so on up to e_T. In order to keep the model simple and stylized, we assume that this productivity endowment sequence is hump-shaped and symmetric (that is, $e_0 = e_T$, $e_1 = e_{T-1}$, $e_2 = e_{T-2}$...) and that it peaks exactly at the middle period of life. We use the following profile:

$$e_s = f(s) = \mu_0 + \mu_1 s + \mu_2 s^2 + \mu_3 s^3 + \mu_4 s^4 \tag{5}$$

such that $f(0) = 0.5$, $f(60) = 0.8$, $f(120) = 1$, $f(180) = 0.8$, and $f(240) = 0.5$. This is a stylized endowment profile which emphasizes that near the beginning and end of the life cycle productivity is low, while in the middle of the life cycle it is high. This endowment profile is displayed in Figure 1. Participant households can sell the productivity units they are endowed with each period on a labor market at a economy-wide competitive wage per efficiency unit.

This particular profile is useful for illustrating the points we wish to emphasize in this paper. The profile suggests that productivity is relatively low at the beginning and end of economic life, but not so low that households might be tempted to supply zero labor in those circumstances. This means that we can restrict attention to interior solutions for the equilibria we study.

2.1.5 Participant household preferences

We denote all variables in real terms except for nominal asset holding a, which is in nominal terms to allow analysis of the NSCNC friction. Accordingly, we let $c_t(t+s) > 0$ denote the date $t+s$ real consumption of the household entering the economy at date t (that is, the date of entry into the economy—the cohort—is indicated by the subscript in this notation), and where $\ell_t(t+s) \in (0,1)$ denotes

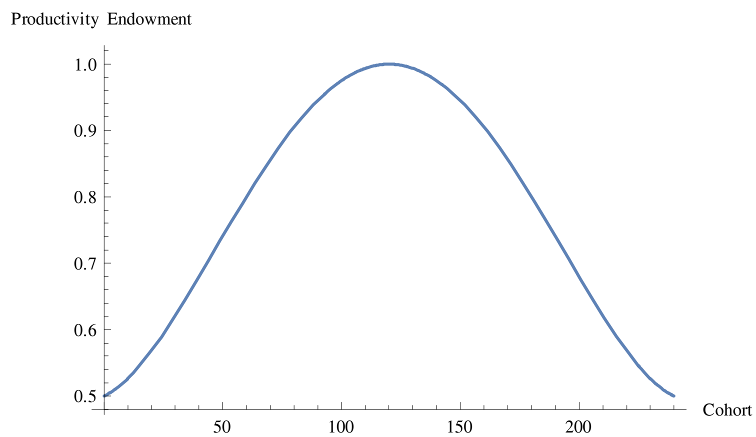

Figure 3: A schematic productivity endowment profile for credit market participant households. The profile is symmetric and peaks in the middle period of the life cycle. The income distribution at each date is this profile multiplied by the real wage. There is considerable income inequality.

the date $t+s$ leisure of the household entering the economy at date t, where the household has one unit of time each period of life. The discount factor $\beta = 1$ and $\eta \in (0,1]$ controls the intratemporal consumption-leisure trade-off. Credit market participant households entering the economy at date t with no asset holdings have preferences given by

$$U_t = \sum_{s=0}^{T} \eta \ln c_t(t+s) + (1-\eta) \ln \ell_t(t+s). \tag{6}$$

There are also credit market participant households that entered the economy at date $t-1$. These households have preferences given by

$$U_{t-1} = \sum_{s=0}^{T-1} \eta \ln c_{t-1}(t+s) + (1-\eta) \ln \ell_{t-1}(t+s). \tag{7}$$

These households will also have a net asset position which we denote by $a_{t-1}(t-1)$, which indicates the net asset holdings carried into the current period from date $t-1$ by the cohort that entered the economy at date $t-1$. Other household preferences for participant households that entered the economy at dates $t-2$, ..., $t-T$ are defined analogously, with net asset positions $a_{t-2}(t-1)$, ..., $a_{t-T}(t-1)$.

2.1.6 Non-participant productivity endowments and preferences

The non-participant households are precluded from using the credit market. They provide a demand for currency in this economy. These households live for $T+1$ periods like their credit market participant cousins, and we will discuss them in terms of their stage of life $0,1,2,...,T$. Their productivity endowment pattern is very different from credit market participant households. In the first period of life they are inactive. Thereafter, in odd-dated stages of life, these households have a productivity endowment $\gamma \in (0,1)$. We think of this as being a low value, and there is no life cycle aspect to it. The cash-using households entering the economy at date t then supply labor inelastically and earn income $\gamma w(t+s)$ for $s = 1,3,5,...,T-1$. In even-dated stages of life, the non-participant households consume. Their period utility is $\ln c_t(t+s)$, $s = 2,4,6,...,T$. These agents work intermittently, carrying the value of their labor income via currency holdings into the period when they wish to consume.

The non-participant household problem generates a conventional currency demand. For brevity, we omit further description of this problem here and refer readers to ABSS (2016). The cash-using households are motivated by an appeal to the unbanked sector of the U.S. economy, which has been estimated to be on the order of 10 to 15 percent of U.S. households.

2.1.7 Technology

The technology is simple extension of the endowment economy idea that "one unit of labor produces one unit of the good," but with appropriate adjustments for productivity endowments e and labor supply $1-\ell_t(t)$. We denote the level of TFP as $Q(t)$. The gross growth rate of Q follows a stochastic process such that

$$Q(t) = \lambda(t-1,t)\,Q(t-1), \tag{8}$$

where $\lambda(t-1,t)$ is the growth rate of productivity between date $t-1$ and date t. The stochastic process driving the growth rate of productivity is $AR(1)$ with mean λ,

$$\lambda(t,t+1) = (1-\rho)\lambda + \rho\lambda(t-1,t) + \sigma\epsilon(t+1) \tag{9}$$

where the unadorned $\lambda > 1$ is the mean growth rate, serial correlation $\rho \in (0,1)$, $\sigma > 0$ is a scale factor, and $\epsilon(t) \sim N(0,1)$.[10] Aggregate output is given by

$$Y(t) = Q(t) L(t). \tag{10}$$

where $L(t)$ is the total supply of labor in this economy.

If we denote $1 - \ell_t(t) \in (0,1)$ as the fraction of participant household time spent working per period, the labor input at date t is given by

$$L^p(t) = e_0 n_t (1 - \ell_t(t)) + e_1 n_{t-1}(1 - \ell_{t-1}(t)) + \cdots + e_T n_{t-T}(1 - \ell_{t-T}(t)) \tag{11}$$

and the supply of labor by non-participants is simply

$$L^{np}(t) = [m_{t-1} + m_{t-3} + \cdots + m_{t-T+1}]\gamma \tag{12}$$

The marginal product of labor is

$$w(t) = Q(t) \tag{13}$$

and we conclude that

$$w(t) = \lambda(t-1,t) w(t-1) \tag{14}$$

as in ABSS (2016). The aggregate output growth rate is then

$$\frac{Y(t)}{Y(t-1)} = \frac{Q(t) L(t)}{Q(t-1) L(t-1)} = \lambda(t-1,t) \frac{L(t)}{L(t-1)}. \tag{15}$$

What is $L(t)/L(t-1)$? We can write

$$\frac{L(t)}{L(t-1)} = \frac{\sum_{s=0}^{T} e_s n_{t-s}(1 - \ell_{t-s}(t))}{\sum_{s=0}^{T} e_s n_{t-s-1}(1 - \ell_{t-s-1}(t-1))} + \frac{m_{t-1}\left[1 + \psi^{-2} + \cdots + \psi^{-T+2}\right]\gamma}{m_{t-2}\left[1 + \psi^{-2} + \cdots + \psi^{-T+2}\right]\gamma}$$

[10] This stochastic process will imply that the zero lower bound on the nominal interest rate can be encountered in this economy, because the shock can always be sufficiently negative. However, a version of the nominal GDP targeting monetary policy can address this situation when $\eta = 1$ (inelastic labor supply) and $\psi = 1$ (no population growth) as discussed in ABSS (2016). We will not address ZLB issues in this paper.

$$\frac{L(t)}{L(t-1)} = \frac{n_t}{n_{t-1}} \frac{\sum_{s=0}^{T} e_s \psi^{-s} (1 - \ell_{t-s}(t))}{\sum_{s=0}^{T} e_s \psi^{-s} (1 - \ell_{t-s-1}(t-1))} + \frac{m_{t-1}}{m_{t-2}} \quad (16)$$

$$= \psi \left[\frac{\sum_{s=0}^{T} e_s \psi^{-s} (1 - \ell_{t-s}(t)) + 1}{\sum_{s=0}^{T} e_s \psi^{-s} (1 - \ell_{t-s-1}(t-1)) + 1} \right]. \quad (17)$$

Our baseline result is that the leisure choices ℓ are independent of the λ shocks and hence of the wage, so they are constants in this formula, meaning in particular that various cohorts will make the same leisure choice at the same stage of the life cycle, represented by $\ell_t(t) = \ell_{t-1}(t-1)$, $\ell_{t-1}(t) = \ell_{t-2}(t-1)$, and so on. This implies that the second term on the right hand side of this expression is equal to unity, leaving the growth in the aggregate labor input equal to the population growth rate alone, that is, $L(t)/L(t-1) = \psi$. We conclude that we can write

$$\frac{Y(t)}{Y(t-1)} = \lambda(t-1,t)\,\psi \quad (18)$$

for the equilibria we wish to study. Along the nonstochastic balanced growth path the gross output growth rate would be $\lambda\psi$. We will show below that the real interest rate equals the real output growth rate period-by-period in the stochastic equilibria we study.

2.1.8 Timing protocol

A timing protocol determines the role of information in the credit sector. We assume that nature moves first and chooses a value for $\epsilon(t)$ which implies a value for the productivity growth rate $\lambda(t-1,t)$ and hence a value for today's real wage $w(t)$. The monetary policymaker moves next and chooses a value for its monetary policy instrument which then implies a value for the price level $P(t)$, as described below. The fiscal policymaker, also described below, moves next and chooses a value nominally-denominated debt $B(t)$. Credit-using households then take $w(t)$ and $P(t)$ as known and make decisions to consume and save via non-state contingent nominal contracts for the following period, carrying a gross nominal interest rate of $R^n(t,t+1)$.

We now turn to describing the public sector portion of this economy.

2.2 The public sector

2.2.1 The fiscal authority

The fiscal authority issues nominally-denominated debt $B(t)$ each period. This publicly-issued nominal debt must compete with the privately-issued debt in the model (issued by relatively young credit market participant households), and so pays the same nominal and real rate of return. We denote the perfectly credible debt-issuance rule of the fiscal authority as

$$B(t) = R^n(t-1,t)\, B(t-1) \tag{19}$$

where $R^n(t-1,t)$ is the gross nominal rate of return on loans in the credit sector of the economy. The fiscal authority is issuing just enough new debt to repay previously-issued debt plus interest. The real value of this debt will be positive when the rate of population growth is positive, $\psi > 1$, and zero if there is no population growth. The public debt here is playing a background role to monetary policy and is modeled after Diamond (1965).

2.2.2 Nominal interest rate contracts

Participant households contract by fixing the nominal interest rate on consumption loans one period in advance. From the participant household (cohort t) Euler equation, the non-state contingent nominal interest rate, $R^n(t,t+1)$, is given by[11]

$$R^n(t,t+1)^{-1} = E_t\left[\frac{c_t(t)}{c_t(t+1)}\frac{P(t)}{P(t+1)}\right]. \tag{20}$$

We call this the contracted nominal interest rate, or simply the "contract rate." The E_t operator indicates that households must use information available as of the end of period t before the realization of $\epsilon(t+1)$. In the equilibria we study, the equity share feature means that all cohorts have the same expectation of their personal consumption growth rates, so that (20) suffices to determine the contract rate. Another way to say this is that there are heterogeneous households in this economy, and in particular some were born at, for instance, date $t-1$. These cohort $t-1$ households would want to contract at the nominal

[11] See Chari and Kehoe (1999) for more details.

rate given by

$$R^n(t,t+1)^{-1} = E_t\left[\frac{c_{t-1}(t)}{c_{t-1}(t+1)}\frac{P(t)}{P(t+1)}\right]. \tag{21}$$

This would similarly be true for all other households entering the economy at earlier dates up to date $t-T+1$. However, in the equilibria we study, it will turn out that

$$\frac{c_t(t)}{c_t(t+1)} = \frac{c_{t-1}(t)}{c_{t-1}(t+1)} = \cdots = \frac{c_{t-T+1}(t)}{c_{t-T+1}(t+1)}, \tag{22}$$

so that these expectations will all be the same and hence (20) suffices to determine the contract rate.[12]

2.2.3 The monetary authority

Equilibrium in the cash market We denote the nominal currency stock issued by the monetary authority as $H(t)$. Consideration of the household problem indicates that there will be $T/2$ of the cohorts demanding currency and that these cohorts will each have income $\gamma w(t)$. The real demand for currency will be given by

$$h^d(t) = [m_{t-1} + m_{t-3} + \cdots + m_{t-T+1}]\,\gamma w(t) \tag{23}$$

$$= m_{t-1}\left[1 + \psi^{-2} + \cdots + \psi^{-T+2}\right]\gamma w(t). \tag{24}$$

Equality of supply and demand in the currency market means

$$\frac{H(t)}{P(t)} = m_{t-1}\left[1 + \psi^{-2} + \cdots + \psi^{-T+2}\right]\gamma w(t). \tag{25}$$

The central bank chooses the rate of currency creation between any two dates $t-1$ and t, $\theta(t-1,t)$, as

$$H(t) = \theta(t-1,t)\,H(t-1). \tag{26}$$

This implies

$$m_{t-1}\left[1 + \psi^{-2} + \cdots + \psi^{-T+2}\right]\gamma w(t)\,P(t)$$
$$= \theta(t-1,t)\,m_{t-2}\left[1 + \psi^{-2} + \cdots + \psi^{-T+2}\right]\gamma w(t-1)\,P(t-1) \tag{27}$$

[12] As an alternative to the standard channels of monetary policy transmission based on nominal rigidities, recent studies have explored this monetary policy transmission via other channels such as segmented financial markets and heterogeneous asset holdings of households. See for example Alvarez and Lippi (2012), Sterk and Tenreyro (2016) among others.

or

$$\theta(t-1,t)=\psi\frac{P(t)}{P(t-1)}\frac{w(t)}{w(t-1)}. \tag{28}$$

The timing protocol implies that $P(t-1)$, $w(t-1)$, and, because nature moves first, $w(t)$, are all known to the policy authority at the moment when θ is chosen. The choice of θ will therefore determine $P(t)$. We conclude that the central bank can in effect choose the date t price level directly under the assumptions we have outlined.

We also assume that the revenue from seigniorage is rebated lump-sum to even-dated cash-using households. This means that there will be no distortion in the cash sector.

This is sufficient to characterize the equilibrium for the cash sector of the economy.

The complete markets policy rule We now assume that the monetary policymaker uses the ability to set the price level at each date t to establish a fully credible policy rule $\forall t$. This policy rule is a version of the one discussed in ABSS (2016) and is given by

$$P(t+1)=\frac{R^{n}(t,t+1)}{\psi\lambda^{r}(t,t+1)}P(t). \tag{29}$$

The term $R^{n}(t,t+1)$ is the contract nominal interest rate effective between date t and date $t+1$, which is an expectation as described above. The term $\lambda^{r}(t,t+1)$ is the ex post realized rate of productivity growth between date t and date $t+1$, that is, the realization of the growth rate for λ observed at date $t+1$. This rule delivers an inflation rate of zero on average. Because $\epsilon(t+1)$, the realized value of the shock, appears in the denominator, this rule calls for countercyclical price level movements. This is a hallmark of nominal GDP targeting as discussed in Sheedy (2014) and Koenig (2013).

3 Equilibrium

3.1 Overview

The equilibrium in this economy that we wish to focus on can be described as follows. We ignore the zero lower bound and we will assume interior solutions for labor supply throughout the paper. The central bank will control the price level according to the perfectly credible policy rule given by (29), creating a doubly-infinite sequence of price levels $\{P(t)\}_{t=-\infty}^{\infty}$. Given this $P(t)$ sequence, the cash-using segment of the economy will clear as described above. The credit-using segment of the economy, which faces a NSCNC friction, will then clear at a gross real interest rate $R(t-1,t)$ at every date. We will conjecture and verify that $R(t-1,t) = \psi\lambda^{r}(t-1,t)$ is both consistent with the optimal solution of each of the participant household problems under the NSCNC friction, and also clears the market for consumption loans. This sequence of real interest rates is also doubly infinite.

We now turn to describing the friction that the credit-using households face, followed by a discussion of the participant household problem, followed by a discussion of market-clearing in the consumption loan market.

3.2 The participant household problem

Here we discuss the solution to the participant household problem.

The participant household maximization for the household entering the economy at date t is given by

$$\max_{\{c_t(t+s),l_t(t+s)\}_{s=0}^{T}} E_t U_t = E_t \sum_{s=0}^{T} \eta \ln c_t(t+s) + (1-\eta) \ln \ell_t(t+s). \tag{30}$$

The credit market participant households will not hold currency in equilibrium because it will be dominated in rate of return. Note however, in equilibrium the publicly-issued debt and privately-issued debt will pay the same rate of return.

We have placed the derivation of key results in the Appendix. Here we will briefly describe how we proceed. Let's begin with the problem of a participant household entering the economy at date t. This household faces a stochastic optimization problem with a finite sequence of budget constraints. This sequence of budget constraints can be combined into a single lifetime budget constraint. If we substitute the monetary policy rule (29) into this lifetime budget constraint, the household's problem is rendered deterministic. The monetary policymaker is credibly promising to offset the productivity shock each period of the participant household's life in such way that, from the individual household's point of view, there is no uncertainty. This problem can then be solved analytically, as is shown in the Appendix.

The solution to the household problem gives a state-contingent plan for consumption and leisure choices. For consumption, the plan is described by

$$c_t(t+s) = \left[\psi^s \prod_{j=0}^{s-1} \lambda^r (t+j, t+j+1)\right] c_t(t). \tag{31}$$

Here $\lambda^r(t, t+j)$ is the ex post realized valued of the productivity growth rate. The state-contingent plan is that the individual's consumption growth rate will equal the realized real output growth rate of the economy. The level of consumption of date t cohort of participant household will be

$$c_t(t) = w(t) \frac{\eta}{T+1} \sum_{j=0}^{T} \psi^{-j} e_j. \tag{32}$$

This, like all key quantities in this model, is linear in the real wage $w(t)$. In the special case when $\eta = 1$ (inelastic labor supply) and $\psi = 1$ (no population growth) this expression indicates that initial consumption should be the cohort share $(1/(T+1))$ of total real income in the credit sector of the economy at date t, which is $w(t) \sum_{s=0}^{T} e_s$. Other values for η and ψ make adjustments for the desirability of leisure and the relative size of different cohorts.

There are other households that entered the economy at earlier dates. These households will solve a similar problem to the one faced by a household in the date t cohort, except that they will carry net assets into the period and they will solve a problem with a shorter horizon. The solution to this problem is similar as shown in the Appendix.

A key question is whether consumption is equalized across households given these household-optimal solutions. If each credit-using household alive at date t is consuming the same amount, then we can say each household has an "equity share" in the real output of the credit sector at date t. Equity share contracting is optimal given homothetic preferences. The Appendix shows that household consumption is in fact equalized in the credit sector and therefore each household has an equity share in the output produced. The proposed monetary policy has completely mitigated the NSCNC friction and restored optimal contracting in the credit sector.

A final step is to ask whether the conjectured sequence of real rates of return (equal to the realized output growth rate in the economy) clears the loan market in each period. In the Appendix we show that this condition is also met, and so we conclude that the conjectured equilibrium is verified.

3.3 The nature of the equilibrium

The stochastic equilibrium verified can be easily characterized in terms of asset-holding, consumption, income, and labor supply. Let's first discuss asset-holding, consumption and income. We will then turn to labor supply in the next subsection.

These figures are generated using the productivity endowment profile (5) described above. In addition, we have set $w(t) = 1$ for all time periods, $\eta = 0.5$ and $\psi = 1$. We will discuss population growth shortly.

A schematic of net asset holding in equilibrium is given in Figure 4. This figure can be viewed as a cross section of net asset holding in the economy at date t. Cohorts are arrayed along the horizontal axis from youngest to oldest, that is, those entering the economy in the current period versus those entering the economy 240 periods in the past. The net asset position is on the vertical axis. Cohorts before the peak earning date at the middle of life are net borrowers and so have negative net positions in the figure, while those after the middle period of life are net lenders. The positive and negative areas of the S-shaped curve sum to zero. Net asset holding is exactly zero in the middle period of life due to the symmetry assumptions underlying the model. The peak borrowers are those at period 60 in the life cycle, which corresponds approximately to age 35; we think of this as schematically representing households wishing to take on mortgages to move housing services consumption forward in the life cycle. The peak lenders are those at period 180 in the life cycle, which corresponds approximately to age 65. We think of this as schematically representing households on the verge of retirement. Net asset-holding positions are linear in the real wage $w(t)$, which is itself stochastic. As the real wage rises, the negative and positive net asset positions increase in proportion to the change in the real wage, in such a way as to keep the positive and negative areas summing to zero each period. As

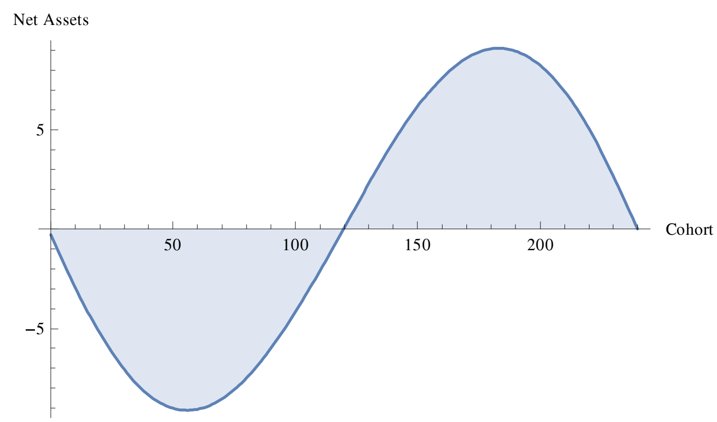

Figure 4: Net asset holding by cohort.

discussed in section 2.2.1, since there is no population growth in this example, government debt equals zero.

There is considerable financial wealth inequality in this economy. If the figure were perfectly triangular, meaning that the two areas were triangles, then 25 percent of the households would hold 75 percent of the assets. The actual figure is close to this.

We have already concluded that consumption by household is exactly equal in the credit sector of the economy, and therefore that each participant household has an equity share in the real output of the credit sector, and furthermore that this represents optimal contracting in the credit sector. This is the basis for our argument that the policy rule (29) provides for optimal monetary policy in this economy.[13] This is illustrated in Figure 5, which shows real income in the economy and real consumption in the economy by cohort.

In the figure, best interpreted as a cross section, cohorts are again arrayed from youngest to oldest on the horizontal axis. Real income by cohort, the hump-shaped line in the figure, is simply the productivity endowment profile multiplied by the real wage at date t. Consumption by cohort, in contrast, is a flat line. This indicates that the credit arrangements in the economy, in conjunction with the monetary policy characterized by the policy rule (29), are

[13] This is combined with the lump-sum transfers to the cash sector of the economy to avoid distortions there. See ABSS (2016) for more details.

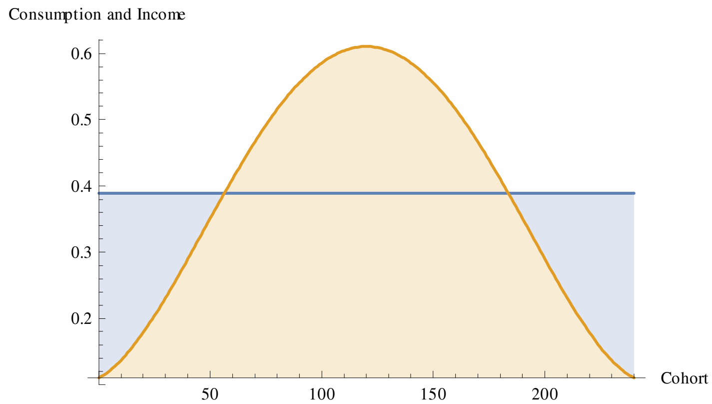

Figure 5: Consumption and income by cohort.

allowing the households in the credit sector to completely mitigate the NSCNC friction. Consumption and income are both linear in the real wage, which is itself stochastic. As the real wage increases stochastically over time, the curve representing income in the economy would increase proportionately, and the flat line representing consumption would increase but remain flat each period. This is consistent with the idea that households in the credit sector would split the income produced in that sector equally at each date, but that, as the technology improves, there would be more output at each date.

3.4 Labor supply in the first best allocation

A key result in this paper is that under the proposed monetary policy rule (29), the labor supply choices, $1 - \ell_t (t + s)$, of participant households depend on demographics alone and are independent of the productivity shock and the real wage. These households supply labor based on their stage in the life cycle, and so the various cohorts do offer differing amounts of labor, but those differing amounts are not dependent on the realization of the shock in any particular period.

The Appendix shows that the leisure choices of a household in cohort t can be characterized as

$$\ell_t (t + s) = \frac{1 - \eta}{\eta} \frac{c_t (t)}{w (t) \psi^{-s} e_s}, \tag{33}$$

for $s = 0, 1, ..., T$. If we substitute in the expression (32) for $c_t (t)$, the real

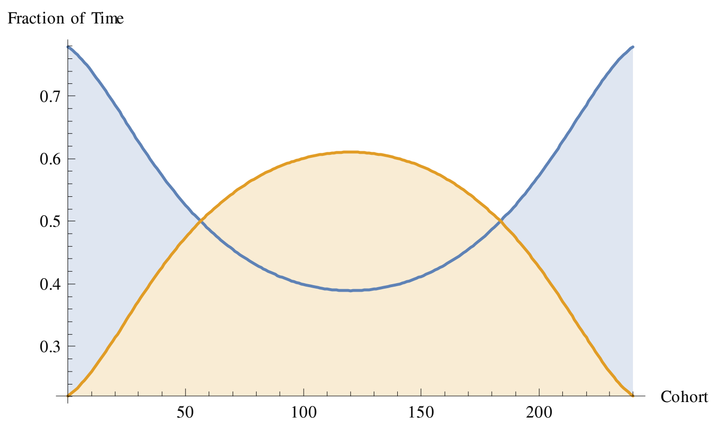

Figure 6: Time devoted to leisure, the u-shaped curve, and work, the hump-shaped curve, by cohort. These curves depend on demographic factors alone.

wage cancels and the choices depend on the parameters η, ψ, and e alone. This expression is given by

$$\ell_t\,(t+s) = \left(\frac{1-\eta}{T+1}\right)\left(\frac{1}{\psi^{-s}e_s}\right)\sum_{j=0}^{T}\psi^{-j}e_j \tag{34}$$

Figure 6 illustrates the leisure choices and labor supply by cohort for the productivity profile (5) with $\eta = 0.5$ and $\psi = 1$. The figure is best viewed as a cross section. Cohorts are arranged from youngest to oldest along the horizontal axis as in previous figures. The vertical axis is the fraction of the one unit time endowment allowed to each household each period which is devoted to leisure. The hump-shaped line in the figure represents the time each cohort devotes to market work, while the inverted u-shaped line represents the fraction of time each cohort spends on leisure. The two lines sum to one for each cohort. The households that are working the most are the ones at the exact middle period (=120) of the life cycle, the peak earners, approximately age 50. The households near the beginning and end of the life cycle devote relatively little time to market work, only about 20 percent versus the 60 percent devoted by the peak income earners. We interpret this as representative of the idea that labor force participation is known to be relatively low near the beginning and end of the life cycle and relatively high during peak earning years. The leisure curve reaches relatively high levels in this figure, but not all time is devoted to leisure even at the beginning and end of the life cycle. This is because we have restricted attention to interior solutions for the purposes of this paper.

The key element of Figure 6, the labor supply figure, from those representing consumption, income, and asset holding, is that labor supply does not depend on the real wage. We interpret this as being consistent with a traditional interpretation of labor supply as discussed at the outset of this paper, in which the labor force participation of households depends most importantly on demographic factors which are unlikely to be influenced by monetary policy. In effect, this makes variables like labor force participation acyclical. In fact, in the economy of this paper, monetary policy is conducted optimally via the policy rule (29), which, in turn, enables households to make optimal labor supply choices independently of productivity shocks hitting the economy. One could view the optimal monetary policy here as allowing households to make labor supply decisions optimally without having to self-insure by adjusting labor supply in response to shocks.

3.5 Comparative statics for an aging population

We have included population growth in this economy. Because we have also maintained an assumption of interior solutions for labor supply (that is, no retirement), population growth is the same as labor force growth. Constant labor force growth ψ enters directly in the expression for the equilibrium real interest rate for this economy. This makes sense because in the life cycle model, a critical feature is the extent of productive capacity of older cohorts, the lenders, versus younger cohorts, the borrowers. Empirically, the changing age profile of an economy is known to be related to changes in key macroeconomic quantities, including real interest rates and real output growth rates.

We can also ask a key question that has been raised in the current monetary policy debate—namely, what is the effect of a slowing rate of population and labor supply growth on the labor supply patterns in the economy? In addition, what, if anything, should monetary policy do about these effects?

We have assumed a constant population growth rate ψ. We can compare the equilibrium of an economy in this class with a relatively high population growth rate to another one with a relatively low population growth rate. The economy with the slower population growth rate will have a lower real rate of interest on average, according to the findings so far.[14] But how will labor supply be affected?

Consider a simple case where $T = 2$, meaning that there are just $T + 1 = 3$ cohorts in the economy. Since we have closed form solution for labor supply, we find that the derivative of the labor supply of the youngest cohort with respect to population growth is positive. That is, when comparing an economy with a relatively high population growth rate to an economy with a relatively low population growth rate, all else equal, the youngest generation would supply less labor in the economy with the lower population growth rate. Similarly, if we analyze the labor supply of the oldest cohort, the sign on the derivative with respect to population growth is negative. That is, older cohorts can be expected to supply more labor in an economy with slower population growth. We omit the calculation here for brevity

The U.S. labor force growth rate has generally been declining during the last two decades. The comparative statics described suggest that we should expect younger cohorts to decrease their labor supply and older cohorts to increase their labor supply in response to the change in the labor force growth rate.

[14] Aksoy, Basso, Smith, and Grasi (2015) report the empirical evidence for 21 OECD countries. Using a life-cycle model with demographic transitions and medium-term dynamics, Aksoy et. al. (2015) also find that declining population growth rates and fertility rates reduce output growth and real interest rates across OECD countries.

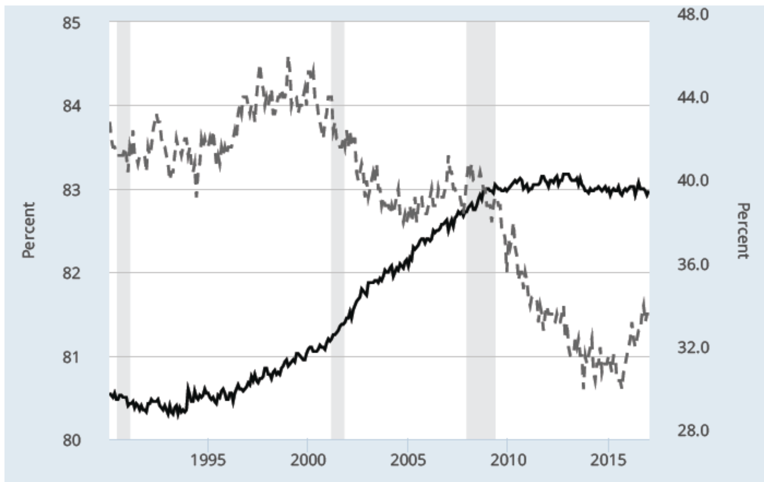

Figure 7: The figure plots the civilian labor force participation rate of 25-54 on the left axis (dash line) and the civilian labor force participation rate of 55 years and over on the right axis (solid line).

Figure 7 displays labor force participation for the U.S. economy by age group over the last two decades. In the model, the midpoint of the life cycle would be about at age 50. The data from the FRED database shows the BLS labor force participation rates, seasonally adjusted, for persons aged 25-54 years and for persons aged 55 and older. In the figure, the time span runs over the last two decades, a period of generally declining population growth rates in the U.S.[15] The participation rate for those aged 25-54 has declined, which corresponds to the prediction for the youngest cohort based on the comparative statics of the $T = 2$ model. The participation rate rises for those aged 55 and older, also corresponding with the prediction of the model for the oldest cohort.[16]

[15] In 1996, the year-over-year population growth rate was about 1.2 percent. By 2016, it had declined to 0.7 percent.

[16] The comparative static result confirmed by the data in Figure 7 helps to resolve a puzzle posed by Hall and Petrosky-Nadeau (2016). In examining U.S. labor force participation data in recent decades, they find that higher income households account for an important part of the decline in participation for households aged 25-54, but not for households over 55. The current model provides an explanation: the over 55 households should be increasing their participation in response to slowing population growth during this period.

We take these results as indicating that some of the same forces that are at work in the model may also be influencing actual labor force participation outcomes in the U.S. economy.[17] While this is far from a comprehensive test, it is encouraging and suggestive that more extensive data-based analysis may be warranted. Life-cycle-based models have been used extensively for this purpose and this one, which includes a monetary friction, could be adapted to provide more convincing comparisons to data.

4 Conclusion

We have provided an analysis of a version of nominal GDP targeting as optimal monetary policy in a model with a credit market friction (NSCNC) following ABSS (2016). The extension here has been to add elastic labor supply as well as population growth. We wanted to study the case of elastic labor supply because, by giving households another margin on which to adjust to shocks, it is possible that nominal GDP targeting would no longer characterize optimal monetary policy as it does in ABSS (2016).

Our main result is that optimal monetary policy is still characterized by a version of nominal GDP targeting even in the expanded setting of this paper. This result could be characterized as a "divine coincidence" result—by providing the optimal monetary policy through countercyclical price level movements, the monetary authority also allows the heterogeneous households to make optimal labor supply decisions. Those decisions do not depend on the aggregate productivity shock in the model and are instead demographically based. We have related these findings to some current issues in U.S. monetary policy.

The life cycle framework used here is quite flexible and has been widely used in the literature to study income and wealth inequality, labor supply issues, as well as consumption and saving. The highly stylized version proposed here and in ABSS (2016) has an important role for monetary policy via a credit market friction as the centerpiece of the theory, similar to Sheedy (2014) and Koenig (2013). While the stylized version provides a lot of clarity as to what

[17] Note that trends in the labor force participation in the euro area are strikingly different from the US, as documented by Nucci and Riggi (2016). See their paper for more details.

is going on in the model and the role for monetary policy, we also think the model is flexible enough to be taken to the data in a more comprehensive way in future versions. In addition, we think that the results here could be expanded to include additional assets, like capital, and also to include idiosyncratic labor income risk, as suggested in Werning (2014).

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5 Appendix

5.1 Details of model solution

The model features heterogeneous households and an aggregate shock, so that the evolution of the asset-holding distribution in the economy is part of the description of the equilibrium. This would normally require numerical computation. However, symmetry, log preferences, and other simplifying assumptions allow solution by “pencil and paper” methods. In this appendix we outline this solution in some detail. A key feature of the solution is that the asset-holding distribution will be linear in the current real wage $w(t)$, and so will simply shift up and down with changes in $w(t)$. Another key feature of the solution will be that the stochastic real rate of return on asset-holding will be equal to the stochastic real output growth rate period-by-period. We do not claim uniqueness of this equilibrium, but we regard the equilibrium we isolate as a natural focal point for this analysis.[18]

We guess-and-verify a solution given a particular price rule for P employed by the monetary authority.

(1) We first propose the state-contingent policy rule for the price level P.

(2a) We then solve the household problem under the proposed policy and determine their state-contingent plan for consumption and non-state contingent plan for leisure.

(2b) Step 2a also applies for all other households entering the economy at earlier dates with shorter horizons and asset holdings from the previous period. We show how these households will adjust their asset holdings as the real wage evolves.

(3) We then establish that per capita consumption is equal, the optimal “equity share” contract.

(4) Finally we verify that under the proposed policy rule and the derived household behavior, the loan market clearing condition is satisfied. This establishes the equilibrium in the credit sector of the economy.

We will assume interior solutions and verify later.

Step 1. The household entering the economy at date t faces uncertainty about income over their life cycle because it does not know what the real wage level is going to be in the future. The proposed state-contingent policy rule

[18] See Feng and Hoelle (2017) for a recent discussion and analysis. Typical quantitative-theoretic applications in the area of stochastic OLG would be unable to address the issues brought out by the Feng and Hoelle (2017) analysis.

eliminates this uncertainty and is given by

$$P(t+1) = \frac{R^n(t,t+1)}{\psi \lambda^r(t,t+1)} P(t) \tag{35}$$

for all t, with $P(0) > 0$.

Step 2a.

First consider households entering the economy at date t. We use

$$\max_{\{c_t(t+s), l_t(t+s)\}_{s=0}^{T}} E_t[\sum_{s=0}^{T} [\eta \ln c_t(t+s) + (1-\eta) \ln \ell_t(t+s)] \tag{36}$$

subject to life-time budget constraint

$$\sum_{s=0}^{T} \left(\frac{P(t+s)}{P(t)} \frac{c_t(t+s)}{\prod_{j=0}^{s-1} R^n(t+j,t+j+1)} \right) \leq \sum_{s=0}^{T} \left(\frac{P(t+s)}{P(t)} \frac{e_s w(t+s)(1-\ell_t(t+s))}{\prod_{j=0}^{s-1} R^n(t+j,t+j+1)} \right). \tag{37}$$

Substitution of the policy rule into the budget constraint for these households yields

$$\sum_{s=0}^{T} \left(\frac{c_t(t+s)}{\psi^s \prod_{j=0}^{s-1} \lambda^r(t+j,t+j+1)} \right) \leq w(t) \sum_{s=0}^{T} \psi^{-s} e_s (1-\ell_t(t+s)). \tag{38}$$

Because $w(t)$ is known by the household at the time when this problem is solved, the uncertainty about future income has been eliminated by the state-contingent policy. The household then solves this problem where μ is the multiplier on the life-time budget constraint. The following sequence of first order conditions for $s = 0, 1, ..., T$ with respect to consumption are

$$\frac{\eta}{c_t(t+s)} = \frac{\mu}{\psi^s \prod_{j=0}^{s-1} \lambda^r(t+j,t+j+1)} \tag{39}$$

which implies

$$c_t(t+s) = \left[\psi^s \prod_{j=0}^{s-1} \lambda^r(t+j,t+j+1) \right] c_t(t) \tag{40}$$

The following sequence of first order conditions for $s = 0, 1, ..., T$ with respect to ℓ are

$$\frac{1-\eta}{\ell_t\,(t+s)} = \mu w\,(t)\,\psi^{-s} e_s \tag{41}$$

Using the FOC for first period consumption to substitute out μ gives the following choices for leisure for $s = 0, 1, ..., T$

$$\ell_t\,(t+s) = \frac{1-\eta}{\eta} \frac{c_t\,(t)}{w\,(t)\,\psi^{-s} e_s} \tag{42}$$

We can then substitute back into the budget constraint, which is

$$(T+1)c_t\,(t) \leq w(t) \sum_{s=0}^{T} \psi^{-s} e_s\,(1 - \ell_t\,(t+s))\,. \tag{43}$$

Substituting for leisure gives

$$(T+1)c_t\,(t) \leq w\,(t) \sum_{s=0}^{T} \psi^{-s} e_s \left(1 - \frac{\eta}{1-\eta} \frac{c_t\,(t)}{w\,(t)\,\psi^{-s} e_s}\right) \tag{44}$$

This is

$$(T+1)\frac{\eta}{\eta} c_t\,(t) + \frac{1-\eta}{\eta} c_t\,(t) = w\,(t) \sum_{s=0}^{T} \psi^{-s} e_s \tag{45}$$

or

$$c_t\,(t) = w\,(t)\,\frac{\eta}{T+1} \sum_{j=0}^{T} \psi^{-j} e_j \tag{46}$$

We conclude that the choice for $c_t\,(t)$, first period consumption depends on today's wage $w\,(t)$ alone. The household has a state-contingent consumption plan for the future. It is to consume

$$c_t\,(t+s) = \left[\psi^s \prod_{j=0}^{s-1} \lambda^r\,(t+j, t+j+1)\right] c_t\,(t)\,. \tag{47}$$

depending on the realizations of future shocks to the TFP growth rate λ.

The amount of leisure chosen at date t and in the future depends on where they are in the life cycle. They are given by

$$\ell_t\,(t+s) = \left(\frac{1-\eta}{T+1}\right)\left(\frac{1}{\psi^{-s} e_s}\right) \sum_{j=0}^{T} \psi^{-j} e_j. \tag{48}$$

If $\eta = 1$ the household will choose no leisure. If $\eta \rightarrow 0$ and $e_0 = e_T$ are small enough, then $\ell_t\,(t)$ and $\ell_t\,(t+T)$ could be larger than one, meaning the households would supply no labor on those dates. This would violate our interior solution assumption. We assume $e_0 = e_T \gg 0$ and η sufficiently large to maintain interior leisure choices.

This household will carry some nominal asset position into the next period. The date t real value of this position is given by

$$\begin{aligned}
\frac{a_t(t)}{P(t)} &= e_0 \left[1 - \ell_t(t)\right] w(t) - c_t(t) && (49) \\
&= e_0 \left[1 - \frac{1-\eta}{T+1} \frac{\left[\sum_{j=0}^{T} \psi^{-j} e_j\right]}{e_0} \right] w(t) - \frac{\eta}{T+1} w(t) \left[\sum_{j=0}^{T} \psi^{-j} e_j\right] && (50) \\
&= w(t) \left[e_0 + \frac{\eta - 1}{T+1} \left[\sum_{j=0}^{T} \psi^{-j} e_j\right] - \frac{\eta}{T+1} \left[\sum_{j=0}^{T} \psi^{-j} e_j\right] \right] && (51) \\
&= w(t) \left[e_0 - \frac{1}{T+1} \left[\sum_{j=0}^{T} \psi^{-j} e_j\right] \right]. && (52)
\end{aligned}$$

Step 2b.

There are also households that entered the economy at date $t-1, t-2, \cdots, t-T$ that would solve a similar problem. These households would have brought nominal asset holdings $a_{t-1}(t-1), a_{t-2}(t-1), \cdots, a_{t-T}(t)$, respectively, into the current period, and have a shorter remaining horizon in their life cycle. Here we will show the solution to a household problem for a household that entered the economy at date $t-1$. In particular, we will show that asset holdings $a_{t-1}(t)$ continues to be linear in the current real wage $w(t)$. We will then infer solutions for all of the other household problems for households entering the economy at dates $t-2, \cdots, t-T$.

The household entering the economy at date $t-1$ solve this problem at date t :

$$\max_{\{c_{t-1}(t+s-1), l_{t-1}(t+s-1)\}_{s=1}^{T}} E_t\left[\sum_{s=1}^{T} \left[\eta \ln c_{t-1}(t+s-1) + (1-\eta) \ln \ell_{t-1}(t+s-1)\right]\right]$$

subject to life-time budget constraint

$$\sum_{s=1}^{T}\left(\frac{P(t+s-1)}{P(t)}\frac{c_{t-1}(t+s-1)}{\prod_{j=0}^{s-1}R^{n}(t+j,t+j+1)}\right)$$

$$\leq\sum_{s=1}^{T}\left(\frac{P(t+s-1)}{P(t)}\frac{e_{s}w(t+s)(1-\ell_{t}(t+s))}{\prod_{j=0}^{s-1}R^{n}(t+j,t+j+1)}\right)+\frac{R^{n}(t-1,t)\,a_{t-1}(t-1)}{P(t)}.. \tag{53}$$

In this "remaining lifetime" budget constraint, we can see from section $2a$ above what the (nominal) value of $a_{t-1}(t-1)$ must have been from last period, namely

$$a_{t-1}(t-1)=P(t-1)\,w(t-1)\left[e_{0}-\frac{1}{T+1}\sum_{j=0}^{T}\psi^{-j}e_{j}\right]. \tag{54}$$

We can therefore find the value of $R^{n}(t-1,t)\,a_{t-1}(t-1)$ as

$$R^{n}(t-1,t)\,a_{t-1}(t-1)=R^{n}(t-1,t)\,P(t-1)\,w(t-1)\left[e_{0}-\frac{1}{T+1}\sum_{j=0}^{T}\psi^{-j}e_{j}\right]. \tag{55}$$

We can use the policy rule $P(t)=\frac{R^{n}(t-1,t)}{\psi\lambda^{r}(t-1,t)}P(t-1)$ and the law of motion for $w(t)$ to simplify the RHS as

$$= R^{n}(t-1,t)\,\frac{P(t)\,\psi\lambda^{r}(t-1,t)}{R^{n}(t-1,t)}\frac{w(t)}{\lambda^{r}(t-1,t)}\left[e_{0}-\frac{1}{T+1}\sum_{j=0}^{T}\psi^{-j}e_{j}\right] \tag{56}$$

$$= \psi P(t)\,w(t)\left[e_{0}-\frac{1}{T+1}\sum_{j=0}^{T}\psi^{-j}e_{j}\right]. \tag{57}$$

Therefore the entire real-valued term is given by

$$\frac{R^{n}(t-1,t)\,a_{t-1}(t-1)}{P(t)} = \psi w(t)\left[e_{0}-\frac{1}{T+1}\sum_{j=0}^{T}\psi^{-j}e_{j}\right] \tag{58}$$

$$= w(t)\left[\frac{T}{T+1}\psi e_{0}-\frac{1}{T+1}\sum_{j=1}^{T}\psi^{-(j-1)}e_{j}\right] \tag{59}$$

Since $w(t)$ and $P(t)$ are known at date t when the consumption-saving decision is made, this is a nonstochastic object. It is linear in $w(t)$. It enters in lump-sum fashion in the budget constraint and so does not affect the first order conditions.

We now substitute the policy rule into the rest of the budget constraint to obtain

$$\sum_{s=1}^{T}\left(\frac{c_{t-1}(t+s-1)}{\psi^{s-1}\prod_{j=0}^{s-1}\lambda^{r}(t+j,t+j+1)}\right)$$
$$\leq w(t)\left(\sum_{s=1}^{T}\psi^{-(s-1)}e_s\left(1-\ell_{t-1}(t+s-1)\right)+\frac{T}{T+1}\psi e_0-\frac{1}{T+1}\sum_{j=1}^{T}\psi^{-(j-1)}e_j\right). \tag{60}$$

As previously there is no income uncertainty for this household.

Let's now turn to the FOC for this problem. We have for $s=1,1,...,T$ with respect to consumption are

$$\frac{\eta}{c_{t-1}(t+s-1)}=\frac{\mu}{\psi^{s-1}\prod_{j=0}^{s-1}\lambda^{r}(t+j,t+j+1)} \tag{61}$$

which implies

$$c_{t-1}(t+s)=\left[\psi^{s}\prod_{j=0}^{s}\lambda^{r}(t+j,t+j+1)\right]c_{t-1}(t) \tag{62}$$

The following sequence of first order conditions for $s=1,...,T$ with respect to ℓ are

$$\frac{1-\eta}{\ell_{t-1}(t+s-1)}=\mu w(t)\,\psi^{-(s-1)}e_s \tag{63}$$

We combine each of these with the corresponding FOC for consumption to give the following choices for leisure for $s=1,...,T$

$$\ell_{t-1}(t+s-1)=\frac{1-\eta}{\eta}\frac{c_{t-1}(t)}{w(t)\,\psi^{-(s-1)}e_s} \tag{64}$$

We can now substitute back into the "remaining life" budget constraint. This yields

$$Tc_{t-1}(t) \leq w(t)\left(\sum_{s=1}^{T}\psi^{-(s-1)}e_s\left(1-\frac{1-\eta}{\eta}\frac{c_{t-1}(t)}{w(t)\psi^{-(s-1)}e_s}\right)+\frac{T}{T+1}\psi e_0-\frac{1}{T+1}\sum_{j=1}^{T}\psi^{-(j-1)}e_j\right) \tag{65}$$

$$T\frac{\eta}{\eta}c_{t-1}(t)+T\frac{1-\eta}{\eta}c_{t-1}(t)=w(t)\sum_{s=1}^{T}\psi^{-(s-1)}e_s+w(t)\left[\frac{T}{T+1}\psi e_0-\frac{1}{T+1}\sum_{j=1}^{T}\psi^{-(j-1)}e_j\right] \tag{66}$$

$$c_{t-1}(t)=w(t)\frac{\eta}{T+1}\left[\psi e_0+\sum_{j=1}^{T}\psi^{-(j-1)}e_j\right]. \tag{67}$$

This is linear in $w(t)$. This household would then have a desired asset position:

$$\frac{a_{t-1}(t)}{P(t)}=e_1w(t)(1-\ell_{t-1}(t))-c_{t-1}(t)+\frac{R^n(t-1,t)\,a_{t-1}(t-1)}{P(t)} \tag{68}$$

$$\begin{aligned}
&= e_1w(t)\left[1+\frac{\eta-1}{\eta}\frac{c_{t-1}(t)}{e_1w(t)}\right]-\frac{\eta}{\eta}c_{t-1}(t)+w(t)\left[\frac{T}{T+1}\psi e_0-\frac{1}{T+1}\sum_{j=1}^{T}\psi^{-(j-1)}e_j\right],\\
&= e_1w(t)-\frac{1}{\eta}c_{t-1}(t)+w(t)\left[\frac{T}{T+1}\psi e_0-\frac{1}{T+1}\sum_{j=1}^{T}\psi^{-(j-1)}e_j\right],\\
&= e_1w(t)-\frac{1}{\eta}w(t)\frac{\eta}{T+1}\left[\psi e_0+\sum_{s=1}^{T}\psi^{-(s-1)}e_s\right]+w(t)\left[\frac{T}{T+1}\psi e_0-\frac{1}{T+1}\sum_{j=1}^{T}\psi^{-(j-1)}e_j\right],\\
&= w(t)\left[e_1-\frac{1}{T+1}\left[\psi e_0+\sum_{s=1}^{T}\psi^{-(s-1)}e_s\right]+\left[\frac{T}{T+1}\psi e_0-\frac{1}{T+1}\sum_{j=1}^{T}\psi^{-(j-1)}e_j\right]\right]\\
&= w(t)\frac{1}{T+1}\left[\psi e_0+e_1-T\sum_{j=2}^{T}\psi^{-(j-1)}e_j\right].
\end{aligned}$$

For all other households at date t who entered the economy at date $t-2, t-3, ..t-T$, consumption and assets at date t will also be linear in $w(t)$.

Step 3.

We now wish to show that per capita consumption is equalized among households. The consumption amounts are

$$c_t(t) = w(t)\frac{\eta}{T+1}\sum_{j=0}^{T}\psi^{-j}ej, \tag{69}$$

$$c_{t-1}(t) = w(t)\frac{\eta}{T+1}\left[\psi e_0 + \sum_{j=1}^{T}\psi^{-(j-1)}e_j\right], \tag{70}$$

$$c_{t-2}(t) = w(t)\frac{\eta}{T+1}\left[\psi^2 e_0 + \psi e_1 + \sum_{j=2}^{T}\psi^{-(j-2)}e_j\right] \ldots.. \tag{71}$$

The cohort entering the economy at date $t-2$ is smaller than the cohort that entered the economy at date t by a factor of ψ^{-2}, and the cohort entering the economy at date $t-1$ is smaller by a factor of ψ^{-1}. Multiplying through by these factors indicates that per capita consumption is equalized under the proposed policy rule for P.

Step 4.

Let's denote the government-issued nominal debt by $B(t)$. It is issued promising a non-state contingent nominal rate of return. The difference between the privately-issued nominal debt and the aggregate nominal asset positions of households must be the government-issued nominal debt. The equilibrium condition is therefore

$$B(t) = \sum_{s=0}^{T-1} n_{t-s}a_{t-s}(t) \tag{72}$$

$$B(t) = R^n(t-1,t)\,B(t-1). \tag{73}$$

The second part of this can be viewed as a bond issuance rule. Today's bond issuance must be just enough to pay principle and interest on previous bond issuance.

The equilibrium conditions can be combined to yield

$$\sum_{s=0}^{T-1} n_{t-s} a_{t-s}(t) = R^n(t-1,t)\left[\sum_{s=1}^{T} n_{t-s} a_{t-s}(t-1)\right] \quad (74)$$

$$n_t\left[\sum_{s=0}^{T-1} \psi^{-s} a_{t-s}\right] = R^n(t-1,t)\left[n_{t-1}\left[\sum_{s=1}^{T} \psi^{-(s-1)} a_{t-s}(t-1)\right]\right] \quad (75)$$

$$\sum_{s=0}^{T-1} \psi^{-s} a_{t-s} = R^n(t-1,t)\left[\sum_{s=1}^{T} \psi^{-s} a_{t-s}(t-1)\right] \quad (76)$$

Substituting out the (nominal) asset positions and noting that

$$\frac{w(t)P(t)}{w(t-1)P(t-1)} = \lambda^r(t-1,t)\frac{R^n(t-1,t)}{\lambda^r(t-1,t)\psi} = \frac{R^n(t-1,t)}{\psi}, \quad (77)$$

the bond issuance rule is satisfied in an equilibrium in which the real rate of interest is equal to the real rate of output growth.