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Abstract

We formulate a flexible generalised hyperbolic (GH) option pricing model, which unlike the version proposed by Eberlein and Prause (2002), has all four of its parameters free to be estimated. We also present six three-parameter special cases: a variance gamma (VG), t, hyperbolic, normal inverse Gaussian, reciprocal hyperbolic and normal reciprocal inverse Gaussian option pricing model. Using S&P 500 Index options, we compare the flexible GH, VG, t and Black-Scholes models. The flexible GH model offers the best out-of-sample pricing overall, while the t special case outperforms the VG for both in-sample and out-of-sample pricing. All three models also improve the orthogonality of implied volatility compared to the Black-Scholes model.

Key words: generalised hyperbolic, t distribution, variance gamma, skewness, Lévy processes

JEL classification: C58, G13

Empirical option prices indicate that the likelihood of extreme logarithmic stock returns is higher than that implied by the Black-Scholes model. Option prices also reveal that market participants pay more to protect themselves from losses than to pursue gains of equivalent magnitude. The statistical implication is that the risk-neutral distribution of log-returns exhibits excess kurtosis and negative skewness (Madan and Milne, 1991; Eberlein and Keller, 1995). These two digressions from the Black-Scholes' normality assumption (Black and Scholes, 1973; Merton, 1973) are in part responsible for its poor empirical pricing results. To combat this deficiency, the generalised hyperbolic (GH) distribution and its six special cases: the variance gamma (VG), t, hyperbolic, normal inverse Gaussian, reciprocal hyperbolic and normal reciprocal inverse Gaussian distributions, can be used to improve option pricing as they accommodate skewness and thicker, semi-heavy tails (Barndorff-Nielsen, 1977).

Eberlein and Prause (2002) proposed a version of a GH option pricing model. However, the accessibility of the model is encumbered by estimation difficulties as one of its four parameters, namely the index parameter, is required to be fixed (Prause, 1999)[1]. Eberlein and Prause assumed that the underlying asset's returns are independent over time. On the other hand, Finlay and Seneta (2012) have also proposed a GH option pricing model that allows for short and long range dependence[2] in squared log-returns. Notwithstanding, this dependent model remains yet to be empirically tested in the literature. In this paper, we focus on addressing the challenges faced by GH option pricing in the independent setting. Our contribution to this field is an unrestricted form of the GH model, the flexible GH option pricing model, where all four parameters are free and can conveniently be estimated. In regard to our flexible GH model's special cases, we present six three-parameter option pricing models. With the exception of the VG option pricing model, which was proposed by Madan, Carr, and Chang (1998), the remaining five models' parameterisations are innovations of this paper.

To construct the flexible GH option pricing model and its special cases, we generalise the Black-Scholes model through the method of subordination (Clark, 1973). The Black-Scholes model's Brownian motion with drift for log-returns is subordinated by a stochastic time-change process. The stochastic time-change follows a generalised inverse Gaussian (GIG) or a special case of the GIG process (Barndorff-Nielsen and Halgreen, 1977). The resulting family of flexible GH processes are pure-jump, infinite activity processes (Barndorff-Nielsen, Mikosch, and Resnick, 2001)[3]. The infinite activity property allows a GH process to move an unlimited number of jumps within an infinitesimally small interval, assisting the model to capture both discrete and continuous asset price movements (Daal and Madan, 2005). The class of flexible GH models also have stochastic drifts and stochastic variances. Though, unlike other stochastic volatility models such as Heston's model (1993), there are no mean-reversion or other time-series dynamics ascribed to the stochastic variances.

In this paper, we also conduct an empirical comparison of the flexible GH, VG, t and Black-Scholes option pricing models using S&P 500 Index options. Of the six flexible GH special cases, we examine the VG and t models because they were not studied by Eberlein and Prause (2002), who instead focused on hyperbolic and normal inverse Gaussian subclasses of their GH option pricing model. Furthermore, a direct comparison of the VG and t models is theoretically motivated since they are complementary special cases of the flexible GH model, under certain conditions elaborated on in Subsection 2.2. In the empirical analyses therefore, we not only monitor which of the four models is superior, but also lend

[1] Prior to Eberlein and Prause's 2002 paper, details about the GH option pricing model and its estimation can be found in Prause (1999), chapter 2.

[2] The option pricing models for assets with long range dependence were pioneered by Heyde (1999), and Heyde and Liu (2001). Special cases of the GH distribution have since been employed in short and long range dependence models, such as the t distribution by Heyde and Leonenko (2005), Finlay and Seneta (2006), and Leonenko, Petherick, and Sikorskii (2011), the VG distribution by Finlay and Seneta (2006), and Leonenko, Petherick, and Sikorskii (2012b), and the normal inverse Gaussian distribution by Leonenko, Petherick, and Sikorskii (2012b,a).

[3] The pure-jump and infinite activity property of the GH process and special cases is a distinguishing property from Heston's model (1993), the Black-Scholes model and from other seminal models, such as Merton's jump-diffusion model (1976), Bates' model (1996), and Pan's model (2002), as noted by Carr and Wu (2004). As a purely discontinuous process, the GH process is also different from the GH diffusion process of Bibby and Sørensen (1997) and Rydberg (1999).

particular attention to the contest between the VG and t special cases.

Our empirical study assesses the option pricing models based on three yardsticks: in-sample fit, the models' misspecifications, and out-of-sample pricing error. We analyse the models' pricing performances for options of all strikes and maturities, in addition to scrutinising their disaggregated pricing results. That is, we investigate the fit of option prices of different strike-to-spot price ratios (moneyness) and different times-to-maturity. First for in-sample fitting, we find that the t model performs as well as the flexible GH model overall. The difference between the two models is encapsulated by the flexible GH model fitting at-the-money options better, and the t model fitting the left tail of the log-return distribution better, as evinced by the t model's superior fit of in-the-money put options. Compared to the VG model, the t model outperforms for all option types. The t model's superiority over the VG model is further verifiable by the flexible GH model's empirical parameter estimates. The flexible GH model more often estimates parameter values that reduce it to the t special case than to the VG special case. Secondly, an orthogonality test of implied volatility to moneyness, time-to-maturity and the interest rate demonstrates that all three models reduce the misspecification inherent in the Black-Scholes model (Rubinstein, 1985). Finally, for out-of-sample pricing, the flexible GH model achieves the lowest pricing error, followed by the t, VG and then the Black-Scholes model. Between the VG and t models, the t model is superior for the majority of moneyness and time-to-maturity combinations.

Our paper is structured as follows. Section 1 begins with a description of a subordinated process and presents a corresponding option pricing framework. In Section 2, we formulate the flexible GH option pricing model and its six special cases, including the two limiting cases. Section 3 comprises our empirical study. The data description, parameter estimates, in-sample fit, orthogonality test and out-of-sample pricing results are provided therein. The conclusion is presented in Section 4.

1 Subordinated Option Pricing Models

In this section, we describe the methodology of constructing an option pricing framework for subordinated stochastic processes. Suppose that logarithmic stock returns, in keeping with the Black-Scholes model, follow a Brownian motion with drift but that as a generalisation of the Black-Scholes model, the log-return process moves at uneven, randomised time-intervals. Such a log-return process is known as a subordinated stochastic process in that it is subordinated to the Brownian motion by the randomised time-change process (Bochner, 1955; Feller, 1966). The concept of randomised time may be interpreted as the passage of intrinsic or economic, rather than physical time. Added as a feature in financial modelling, it captures for instance, the empirical market characteristics that information 'arrives' at the market at uneven and unpredictable time intervals, or that trading volume fluctuates randomly throughout the trading day (Hurst, Platen, and Rachev, 1997).

We may denote the stochastic intrinsic time process with g_t. Let g_t be a Lévy process (Lévy, 1937), constructed by summing stationary and independent increments, g_1, where $g_1 = g_{t+1} - g_t$ for $t \geq 0$. The intrinsic time increment over a unit of physical time, g_1, is required to follow a non-negative, infinitely divisible distribution with a unit expectation, $\mathrm{E}[g_1] = 1$, for the intrinsic time interpretation to hold. The subordinated stochastic process, X_t, is then formed by introducing a scaled Wiener process, $\sigma W(\cdot)$, and drift, θ, onto the intrinsic time scale of g_t (Clark, 1973) as such:

$$X_t = \theta g_t + \sigma W(g_t), \tag{1}$$

where $\sigma > 0$ and $W(\cdot)$ is independent from g_t. The increment, $X_1 = X_{t+1} - X_t$, will follow the normal mean-variance mixture distribution that results from using the distribution of g_1 as the mixing density[4] (Barndorff-Nielsen, Kent, and Sørensen, 1982). As a normal mixture, the distribution of X_1 will be leptokurtic. That is, it will have heavier tails and a higher peak than the normal distribution (Hurst, Platen, and Rachev, 1997). The distribution will also be skewed, due to the mixing in the mean through θ. The distribution of X_1's infinite divisibility follows from the infinite divisibility of g_1 (Barndorff-Nielsen and Halgreen, 1977), and consequently X_t will be a Lévy process.

The subordinated process, X_t, is adopted into stock price modelling as follows:

$$S_t = S_0 \exp\{(r-q)t + X_t + \omega t\}, \tag{2}$$

where S_t is the spot stock price at (physical) time t, r is the risk-free rate and q is the dividend yield. The risk-neutral drift adjustment, ω, is equal to $-\ln \phi_{X_1}(-i)$, where $\phi_{X_1}(u)$ denotes the characteristic function of X_1. This result for ω is derived in Appendix A, Equation (A.6).

The following subordinated option pricing framework allows for computation of a European call or put option price using the characteristic function of g_1, $\phi_{g_1}(u)$:

$$\begin{aligned}
C_t &= S_t e^{-q\tau}\Pi_1 - Ke^{-r\tau}\Pi_2 \qquad (3)\\
P_t &= Ke^{-r\tau}(1-\Pi_2) - S_t e^{-q\tau}(1-\Pi_1)\\
\Pi_1 &= \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left(\frac{e^{-iw\log(K)}\phi_{\log S_t}(w-i)}{iw\phi_{\log S_t}(-i)}\right) dw\\
\Pi_2 &= \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left(\frac{e^{-iw\log(K)}\phi_{\log S_t}(w)}{iw}\right) dw\\
\phi_{\log S_t}(u) &= S_0 \exp\left\{iu\left[r - q - \ln\phi_{X_1}(-i)\right]t\right\}\ \phi_{X_1}(u)^t\\
\phi_{X_1}(u) &= \phi_{g_1}\left(u\theta + \frac{i\sigma^2 u^2}{2}\right),
\end{aligned}$$

[4]Computation of the option price does not require a closed form of the density function of X_t. As proven in Appendix A, where X_t is a Lévy process, it is sufficient to use the characteristic function. Hence, there is no requirement that distributions of g_1 or X_1 be closed in convolution (Hurst, Platen, and Rachev, 1997).

where C_t and P_t are the call and put option price at time t, $\tau = T - t$ is the time-to-maturity, K is the strike price, and g_1 is the intrinsic time-change over a unit of physical time. The derivation of this result can be found in Appendix A. Existing models that fit within the broader class of subordinated option pricing models, for independent asset returns, include the VG option pricing model, where g_1 follows a gamma distribution (Madan, Carr, and Chang, 1998), and the log-stable option pricing model, under which g_1 follows an $\frac{\alpha}{2}$-stable distribution (Hurst, Platen, and Rachev, 1999). The hyperbolic model of Eberlein, Keller, and Prause (1998) and the GH model of Eberlein and Prause (2002) are not parameterised as subordinated models since under these models, g_1 does not have a unit expectation.

2 The Flexible GH Option Pricing Model and Its Special Cases

2.1 The Flexible GH Model

In this section, we parameterise the GH process as a subordinated process to accord with the subordinated option pricing framework of Section 1. It is the subordinated parameterisation that allows all four of the flexible GH option pricing model's parameters to be estimated. To characterise X_t as a GH process, let g_1 follow a generalised inverse Gaussian (GIG) distribution with parameters p (any real number), $a = \gamma^2 \geq 0$ and $b = \delta^2 \geq 0$. Let us define the parameter $\zeta = \delta\gamma$. Transforming the parameter domains given in Barndorff-Nielsen and Halgreen (1977), the parameters, p and ζ must satisfy either of the following conditions for $\mathrm{E}[g_1]$ and in turn for $\mathrm{E}[X_1]$ to be well defined[5]:

$$\begin{aligned} &\text{(i)} \quad \zeta > 0 \quad \text{if } -1 \leq p \leq 0, \\ &\text{(ii)} \quad \zeta \geq 0 \quad \text{otherwise.} \end{aligned} \tag{4}$$

The GIG distribution and its subclasses are infinitely divisible and non-negative (Barndorff-Nielsen and Halgreen, 1977). To impose the unit expectation condition on g_1, let

$$\gamma^2 = \zeta \frac{K_{p+1}(\zeta)}{K_p(\zeta)}, \tag{5}$$

where $K_h(\cdot)$ is the modified Bessel function of the third kind with index h (Jørgensen, 1982). This result follows from $E[g_1] = \frac{\zeta}{\gamma^2} \frac{K_{p+1}(\zeta)}{K_p(\zeta)} = 1$. The flexible GH model's four parameters are therefore a volatility parameter, σ, a skewness parameter, θ, and two kurtosis parameters, p and ζ, where p is also known as the index parameter. A lower p or ζ leads to a higher kurtosis.

To price an option under the flexible GH model, the subordinated option pricing framework from

[5]The parameter domains given in Barndorff-Nielsen and Halgreen (1977): $\delta \geq 0$, $\gamma > 0$ if $p > 0$, $\delta > 0$, $\gamma > 0$ if $p = 0$, and $\delta > 0$, $\gamma \geq 0$ if $p < 0$, are sufficient for the GIG and GH distributions to be defined. However, the subordinated model in Section 1 further requires the mean of the GIG distribution, $\mathrm{E}[g_1]$, to be well defined. The parameter domains in Equation (4) therefore incorporate the result from Jørgensen (1982), that the GIG distribution will have a well defined mean except where $\gamma = 0$ for $-1 \leq p < 0$.

Equation (3) may be used where the characteristic function of $\log S_t$ is given by

$$\phi_{\log S_t}(u) = S_0 \exp\{iu(r-q+\omega)t\} \left[1 - \frac{2}{\gamma^2}\left(iu\theta - \frac{1}{2}\sigma^2 u^2\right)\right]^{-\frac{pt}{2}} \left[\frac{K_p\left(\zeta\sqrt{1-\frac{2}{\gamma^2}\left(iu\theta - \frac{1}{2}\sigma^2u^2\right)}\right)}{K_p(\zeta)}\right]^t, \tag{6}$$

and $\omega = -\frac{p}{2}\ln\left[\left(1-\frac{2}{\gamma^2}\left(\theta+\frac{1}{2}\sigma^2\right)\right)\right] + \ln\left[\frac{K_p\left(\zeta\sqrt{1-\frac{2}{\gamma^2}\left(\theta+\frac{1}{2}\sigma^2\right)}\right)}{K_p(\zeta)}\right]$ for $\theta < \left(\frac{\gamma^2}{2} - \frac{\sigma^2}{2}\right)$. The model is derived from $\phi_{g_1}(u)$, the characteristic function of the GIG distribution, as given in Appendix B.1.

2.2 Special Cases of the Flexible GH Model

In addition to the flexible GH model, in this section we provide six three-parameter option pricing models, which are special cases of the flexible GH option pricing model. These six models can be divided into two groups: first, the four special cases obtained by restricting p and second, the two special or limiting cases where $\zeta = 0$ (Barndorff-Nielsen, Mikosch, and Resnick, 2001; Barndorff-Nielsen and Shephard, 2012). As to restricting p, where $p = 1$, the flexible GH model reduces to a hyperbolic (H) model[6], which uses a positive hyperbolic (PH) distribution for g_1. Setting $p = -1$ leads to a reciprocal hyperbolic (RH) model, in which g_1 follows a reciprocal positive hyperbolic (RPH) distribution. For $p = -\frac{1}{2}$, we have a normal inverse Gaussian (NIG) distribution with g_1 following an inverse Gaussian (IG) density, and for $p = \frac{1}{2}$, we reach a normal reciprocal inverse Gaussian (NRIG) distribution, which has g_1 following a reciprocal inverse Gaussian (RIG) density. For these four mutually exclusive special cases, imposing the restriction that $\gamma^2 = \zeta\frac{K_{p+1}(\zeta)}{K_p(\zeta)}$ from Equation (5), will satisfy the subordinated model's requirement from Section 1 that $\text{E}[g_1] = 1$.

On the other hand, where $\zeta = 0$, the flexible GH model reduces to two other subclasses. For $\delta = 0$ and $p > 0$, we obtain a VG model[7], which has g_1 following a gamma distribution, $\Gamma(p, \frac{\gamma^2}{2})$. For $\gamma = 0$ and $p < -1$, the flexible GH model reduces to a t model[8] with $-2p$ degrees of freedom[9], under which g_1 follows a reciprocal gamma density, $\text{R}\Gamma(-p, \frac{\delta^2}{2})$. To satisfy $\text{E}[g_1] = 1$ in the VG model, where $g_1 \sim \Gamma(\alpha, \beta)$, $\text{E}[g_1] = \frac{\alpha}{\beta} = \frac{2p}{\gamma^2} = 1$ such that $\gamma = \sqrt{2p}$, for $p > 0$. For the t model[10], where $g_1 \sim R\Gamma(\alpha, \beta)$, $\text{E}[g_1] = \frac{\beta}{\alpha - 1} = \frac{\delta^2}{2(-p-1)} = 1$ such that $\delta = \sqrt{2(-p-1)}$, for $p < -1$. For $\zeta = 0$, the VG and t models are complementary special cases in respect of the flexible GH model's domain for p, under Equation (4). We may note that all six special cases are also skewed, since no restriction is imposed on the parameter

[6] Further, where $\delta = 0$ in the hyperbolic distribution, we retrieve the Laplace distribution (Barndorff-Nielsen, 1977).

[7] The variance gamma distribution is a special case of the normal gamma distribution obtained when the gamma mixing density has equal parameters (Madan and Seneta, 1990; Choy and Chan, 2008).

[8] The t distribution used here is different to other skewed versions of the t distribution that have been proposed by Hansen (1994), Jones and Faddy (2003) and Azzalini and Capitanio (2003). For instance, while these alternative skewed t distributions all have two heavy tails, the skewed t distribution that is a subclass of the GH distribution, as used in this paper, has one heavy tail and one semi-heavy tail (Aas and Haff, 2006).

[9] The degrees of freedom is 2α where $g_1 \sim R\Gamma(\alpha, \beta)$ (Praetz, 1972; Blattberg and Gonedes, 1974).

[10] The normalisation of $\text{E}[g_1] = 1$ means that the t distribution is not a 'Student' t distribution, which requires equal parameters in the reciprocal gamma density of g_1 (Seneta, 2004).

θ. Table 1 summarises the flexible GH model's parameter conditions required to obtain its special cases. The number of three-parameter option pricing models nested by our flexible GH model exceeds that for Eberlein and Prause's GH model (2002), as their fixing of p enables only four three-parameter special cases to be obtained.

Table 1: Special cases of the flexible GH option pricing model.

GH$(p, \zeta, \theta, \sigma)$ **parameter conditions**	**Density of** X_1	**Density of** g_1
Hyperbolic model		Positive hyperbolic
$\mathrm{GH}(1, \zeta \geq 0, \theta, \sigma)$, $\delta \geq 0$, $\gamma > 0$	$\mathrm{H}(\zeta, \theta, \sigma)$	$\mathrm{PH}(\delta, \gamma)$
Reciprocal hyperbolic model		Reciprocal positive hyperbolic
$\mathrm{GH}(-1, \zeta > 0, \theta, \sigma)$, $\delta > 0$, $\gamma > 0$	$\mathrm{RH}(\zeta, \theta, \sigma)$	$\mathrm{RPH}(\delta, \gamma)$
Normal inverse Gaussian model		Inverse Gaussian
$\mathrm{GH}(-\frac{1}{2}, \zeta > 0, \theta, \sigma)$, $\delta > 0$, $\gamma > 0$	$\mathrm{NIG}(\zeta, \theta, \sigma)$	$\mathrm{IG}(\delta, \gamma)$
Normal reciprocal inverse Gaussian model		Reciprocal inverse Gaussian
$\mathrm{GH}(\frac{1}{2}, \zeta \geq 0, \theta, \sigma)$, $\delta \geq 0$, $\gamma > 0$	$\mathrm{NRIG}(\zeta, \theta, \sigma)$	$\mathrm{RIG}(\delta, \gamma)$
Variance gamma model		Gamma
$\mathrm{GH}(p > 0, 0, \theta, \sigma)$, $\delta = 0$, $\gamma = \sqrt{2p}$	$\mathrm{VG}(p, \theta, \sigma)$	$\Gamma(p, p)$
t model		Reciprocal gamma
$\mathrm{GH}(p < -1, 0, \theta, \sigma)$, $\delta = \sqrt{2(-p-1)}$, $\gamma = 0$	$t_{-2p}(-p, \theta, \sigma)$	$\mathrm{R}\Gamma(-p, -p-1)$

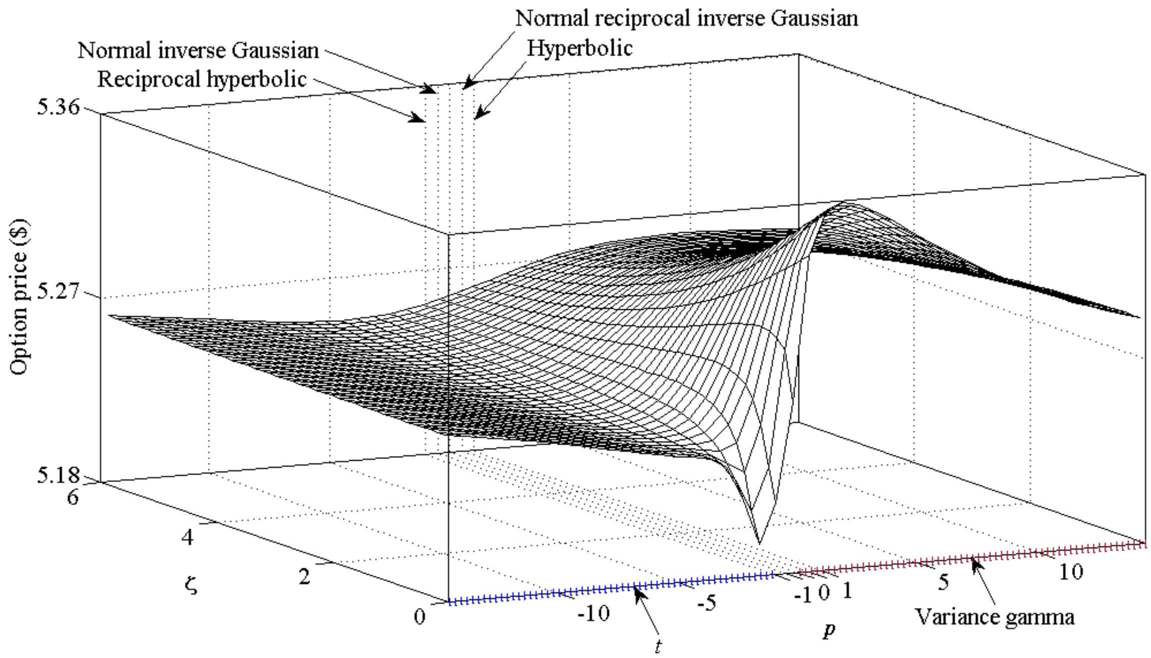
$\zeta = \delta\gamma$. For the hyperbolic, reciprocal hyperbolic, normal inverse Gaussian and normal reciprocal inverse Gaussian models, γ is given by Equation (5). For all models, $\mathrm{E}[g_1] = 1$.

Figure 1 demonstrates the diversity among the six special cases of the flexible GH model. To generate the surface, we simulated flexible GH option prices for varying p and ζ. For a call option with 0.95 moneyness and with three weeks until expiry, the relationship between kurtosis and the option price manifestly differs across the special cases. For $p < 0$, including the NIG, RH and t subclasses, a higher kurtosis (lower p or ζ) corresponds with a lower option price whereas the opposite relationship can be seen when $p > 0$, which encompasses the NRIG, H and VG models. Further, under the VG model, for $p < 2$, the relationship inverts and an increased kurtosis reduces the option price. Albeit only one instance of a variety of surfaces that could be shown for options of alternate strike prices and maturities, Figure 1 can attest to the option pricing flexibility that having two kurtosis parameters affords the proposed GH model compared to its special cases, which each only have one kurtosis parameter.

2.3 The Limiting Cases

For the four GH special cases that result from restricting p, the option price can be computed using the characteristic function of the flexible GH model presented in Equation (6). However, for the special cases where $\zeta = 0$, the limiting case of the modified Bessel function of the third kind with index h, $K_h(\cdot)$, may be used to obtain a parsimonious form of the characteristic function, $\phi_{\log S_t}(u)$. In this subsection we present the limiting GH cases under the VG and t models.

Figure 1: Simulated option prices under the flexible GH model.



The price of an in-the-money call option with a strike price equal to 95% of the spot price, time-to-maturity equal to 3 weeks, $\theta = -0.04$ and $\sigma = 0.17$. The six special cases are reciprocal hyperbolic at $p = -1$, normal inverse Gaussian at $p = -\frac{1}{2}$, normal reciprocal inverse Gaussian at $p = \frac{1}{2}$, hyperbolic at $p = 1$, variance gamma at $\zeta = 0$ and $p > 0$, and the t model at $\zeta = 0$ and $p < -1$. At $\zeta = 0$ and $-1 \leq p \leq 0$, the mean of the GH distribution is undefined.

2.3.1 The VG Model

For the VG model, we can match the flexible GH model's parameters to the existing VG option pricing model's parameters (Madan, Carr, and Chang, 1998). The VG model has three parameters, $\nu > 0$, θ and σ. The parameters θ and σ are equivalent to those in the flexible GH model, while ν in the VG model is equal to $\frac{1}{p}$ under the flexible GH model. Under the VG model, ν is also the variance of the gamma variable, g_1. The VG option price can be computed using the model in Equation (3) together with the characteristic function,

$$\phi_{\log S_t}(u) = S_0 \exp\{iu(r - q + \omega)t\} \left[1 - \nu\left(iu\theta - \frac{1}{2}\sigma^2 u^2\right)\right]^{-\frac{t}{\nu}}, \tag{7}$$

where $\omega = \frac{1}{\nu} \ln\left[1 - \nu\left(\theta + \frac{1}{2}\sigma^2\right)\right]$ for $\theta < \left(\frac{1}{\nu} - \frac{\sigma^2}{2}\right)$. In Appendix B.2, we prove this result, and in Appendix C we show that it is the limiting case of the characteristic function used in the flexible GH model (Equation (6)).

2.3.2 The t Model

A form for an option pricing model that uses the GH special case and skewed version of the t distribution has not yet been proposed in the literature[11]. Using the three parameters $\nu = -\frac{1}{p}$, where $0 < \nu < 1$, θ and σ to mirror the VG model, we propose that the t option price, with $\frac{2}{\nu}$ degrees of freedom, be calculated using

$$\phi_{\log S_t}(u) = S_0 \exp\{iu(r-q+\omega)t\}\times \left[\frac{2\left(\frac{1}{\nu}-1\right)^{\frac{1}{2\nu}}}{\Gamma(\frac{1}{\nu})}\left[-\left(iu\theta-\frac{1}{2}\sigma^2u^2\right)\right]^{\frac{1}{2\nu}} K_{\frac{1}{\nu}}\left(\sqrt{-4\left(\frac{1}{\nu}-1\right)\left(iu\theta-\frac{1}{2}\sigma^2u^2\right)}\right)\right]^t, \quad (8)$$

where $\omega = -\ln\left[\frac{2\left(\frac{1}{\nu}-1\right)^{\frac{1}{2\nu}}}{\Gamma(\frac{1}{\nu})}\left[-\left(\theta+\frac{1}{2}\sigma^2\right)\right]^{\frac{1}{2\nu}} K_{\frac{1}{\nu}}\left(\sqrt{-4\left(\frac{1}{\nu}-1\right)\left(\theta+\frac{1}{2}\sigma^2\right)}\right)\right]$ for $\theta < -\frac{\sigma^2}{2}$. $K_h(\cdot)$ is the modified Bessel function of the third kind with index h. Appendix B.3 provides the proof of this result using the reciprocal gamma characteristic function, $\phi_{g_1}(u)$, and Appendix C shows how it is a limiting case of the flexible GH model. We may note that the t distribution is the only instance of the GH distribution where, rather than having two semi-heavy tails, it has one heavy tail (the tail in the direction of the skewness) and one semi-heavy tail (Aas and Haff, 2006).

3 Empirical Study

3.1 The Data

The data used are S&P 500 Index European options observed in 2012. Whilst we analyse call and put data for the years 2008-2014, results using one year of put options are presented for conciseness. Put options are featured because they carry more information about the left tail of the log-return distribution than call options, corresponding to the side on which the literature has found the log-distribution to be skewed (Madan and Milne, 1991). Data for the year 2012 are chosen because they contain a variety of economic conditions, ranging from bullish to neutral to bearish. The S&P 500 Index during 2012 and its implied volatility index, VIX, can be seen in the lower two panels of Figure 2. Option expiry is monthly, occurring on the third Friday of each month. Option prices with moneyness (the strike-to-spot price ratio) of between 0.94 and 1.06 and times-to-maturity greater than one week are sampled amounting to 60 option prices per day, on average. In total, our sample consists of 250 trading days and 15,058 put option prices, as characterised in Table 2. Finally, the risk-free rate used is the annualised yield of one-month U.S. Treasury bills.

[11] In Yeap (2014), a version of the t option pricing model was proposed, the Skew-t option pricing model, which differs to the model presented in this paper only in that $\nu = \mathrm{Var}[g_1]$. In this paper, we revised the parameterisation to allow for the t model to be a special case of the flexible GH model. The variance of g_1 is now given by $\frac{\nu}{1-2\nu}$ instead. Otherwise, in the independent asset returns setting, a symmetric t option pricing model has been proposed (Cassidy, Hamp, and Ouyed, 2010).

Table 2: Options data characteristics.

	Short-term 1 week $< \tau <$ 1 month	**Medium-term** 1 month $\leq \tau \leq$ 3 months	**Long-term** $\tau >$ 3 months	**All**
Out-of-the-money	8%	24%	11%	43%
$0.94 < \frac{K}{S} < 0.98$	[$3.49]	[$13.40]	[$26.86]	[$14.87]
At-the-money	8%	22%	11%	41%
$0.98 \leq \frac{K}{S} \leq 1.02$	[$16.53]	[$29.20]	[$43.80]	[$30.55]
In-the-money	4%	9%	3%	16%
$1.02 < \frac{K}{S} < 1.06$	[$51.81]	[$58.79]	[$71.32]	[$59.01]
All	20%	55%	25%	100%
	[$18.29]	[$27.48]	[$39.17]	[$28.42]

A contingency table for S&P 500 Index European put options observed during 2012. $n = 15,058$. Average option prices are given in square brackets. K is the strike price, S is the spot price, and τ is time-to-maturity.

3.2 Parameter Estimation and In-Sample Fit

In our empirical study, we compare the flexible GH option pricing model to the VG, t and Black-Scholes models. Of the six three-parameter special cases, we focus on the VG and t models because previously the fixing of p precluded the GH option pricing model from reducing to the VG and t models (Prause, 1999). For brevity, our flexible GH option pricing model is referred to as just the GH model in the empirical sections to follow. First, we examine in-sample fit. Second, we carry out misspecification diagnostics. Thirdly, we evaluate the four models' out-of-sample pricing errors.

Before reporting the three performance metrics, we commence with a discussion of the risk-neutral parameter estimates. The objective function is the sum of squared percentage pricing errors[12] and the optimisation is conducted daily[13], instructed by a Nelder-Mead simplex algorithm (Gilli and Schumann, 2012). From Table 3 and starting with the volatility parameter, σ, the Black-Scholes model estimates the smallest average σ equal to 14.6%, compared to the VG model (15.0%) and the t and GH models (both 15.8%). The t model estimates the single largest σ equal to 24.8%. For the skewness parameter, θ, on all days θ is negative. Average θ values are similar across models: the VG model, -0.044, t model, -0.047 and GH model, -0.046. However, the t model estimates the most negative θ equal to -0.149, compared to the most negative θ for the VG model equal to -0.100 and the GH model, -0.098. The top two panels of Figure 2 show a contemporaneity between higher σ estimates and more negative θ estimates[14], such as during mid-May to mid-June 2012. This behaviour of σ and θ suggests a detection of the leverage effect (Christie, 1982), where volatility is higher when log-returns are more negatively skewed.

[12] A similar objective was used by Madan, Carr, and Chang (1998) and accords with investors' interest in rate of return rather than absolute option price changes.

[13] The sample size varied from day to day but on average was 60, which is consistent with the literature. In Bakshi, Cao, and Chen (1997), daily estimation involved on average 52 options per sample.

[14] The σ and θ estimates under the VG and t models were not superimposed on Figure 2 since their relationships were similar to that under the GH model.

Table 3: Empirical risk-neutral parameter estimates.

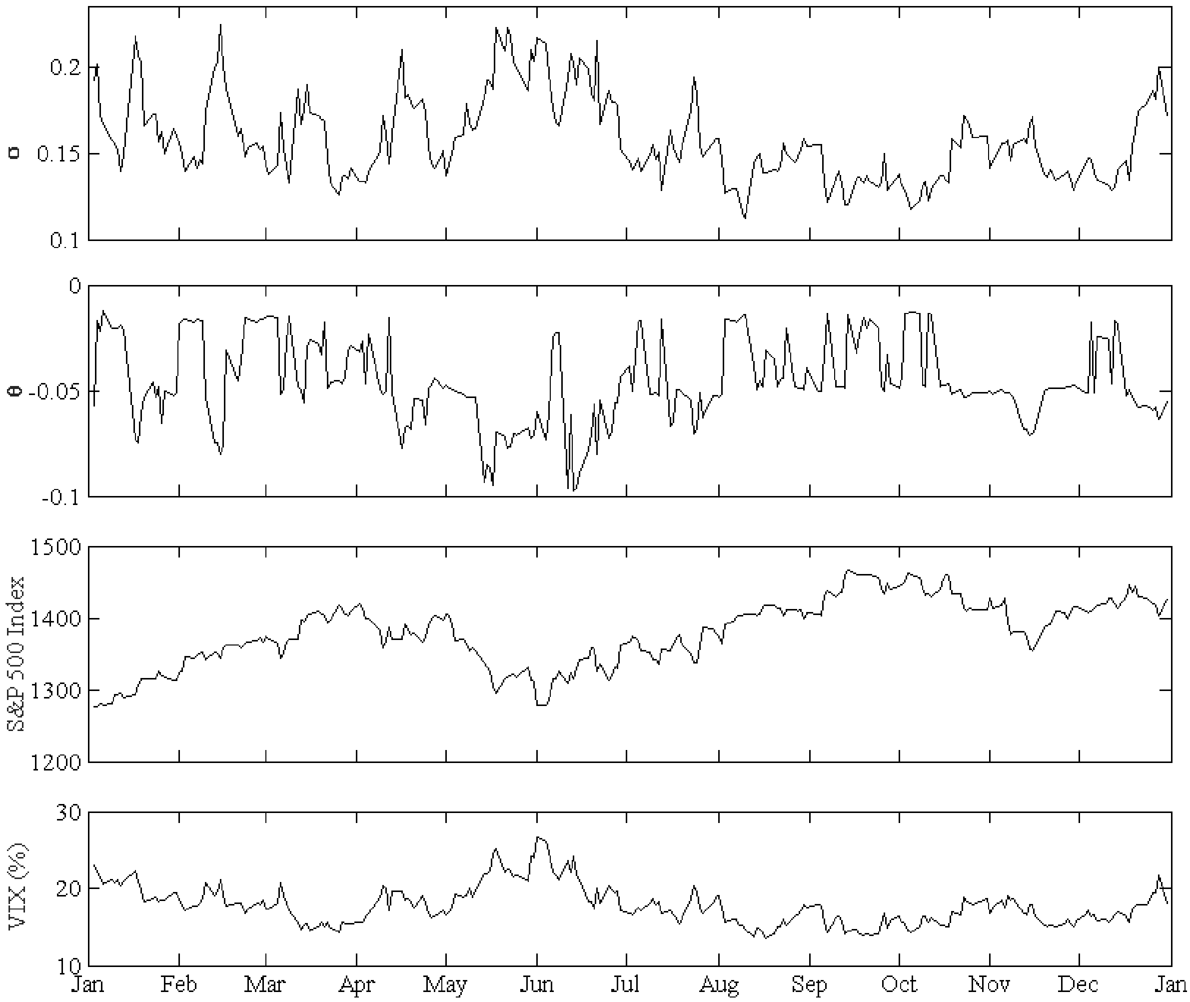
Parameter	**Mean**	**Standard deviation**	**Minimum**	**Maximum**
Black-Scholes				
σ	0.146	0.017	0.116	0.206
RMSPE	16.69%	4.49%	8.85%	34.65%
VG model				
σ	0.150	0.022	0.114	0.224
θ	-0.044	0.016	-0.100	-0.009
p	9.305	9.752	1.777	62.350
RMSPE	9.50%	3.81%	4.42%	26.52%
t **model**				
σ	0.158	0.025	0.113	0.248
θ	-0.047	0.021	-0.149	-0.013
p	-5.550	5.569	-42.824	-1.448
RMSPE	8.38%	3.28%	3.51%	23.98%
GH model				
σ	0.158	0.025	0.113	0.225
θ	-0.046	0.020	-0.098	-0.012
p	-5.151	5.465	-30.180	8.014
ζ	0.417	1.483	0.000	18.778
RMSPE	8.38%	3.27%	3.51%	23.98%

Parameter estimates and RMSPE (root mean squared percentage error) are daily averages for the year 2012, which includes 15,058 put prices over 250 days. The average sample size is therefore 60.

For the kurtosis parameter, p, the VG model's average estimate of 9.305 is higher in magnitude than the average p estimate under the t model, which equals -5.550 (11.1 degrees of freedom). With respect to the models' maximum magnitudes of p, the VG model obtains 62.350, while the t model obtains 42.824, and the GH model, 30.180. On the other hand, the models' lowest magnitudes for p are similar: the VG model, 1.777, the t model, 1.448 and GH model, 1.117. The minimum magnitude of p under the GH model indicates that the GH model does not estimate a log-distribution that corresponds precisely to the reciprocal hyperbolic ($p = -1$), NIG ($p = -\frac{1}{2}$), NRIG ($p = \frac{1}{2}$) or hyperbolic ($p = 1$) special case on any of the days sampled. Elucidating a separate stylised fact about options data, the time-series view of p in Figure 3, also reveals cyclical behaviour in the tail thickness of the log-distribution. The tails begin the trading month[15] thick (low magnitude of p) and then become thinner, with the magnitude of p culminating on the second Friday of each month. Then, drastically, p falls in magnitude and the log-distribution is fat-tailed in the week through to expiry.

[15]The options trading month is in accordance with the option expiry schedule. Expiry is on the third Friday of each calendar month and so the trading month begins on the subsequent Monday.

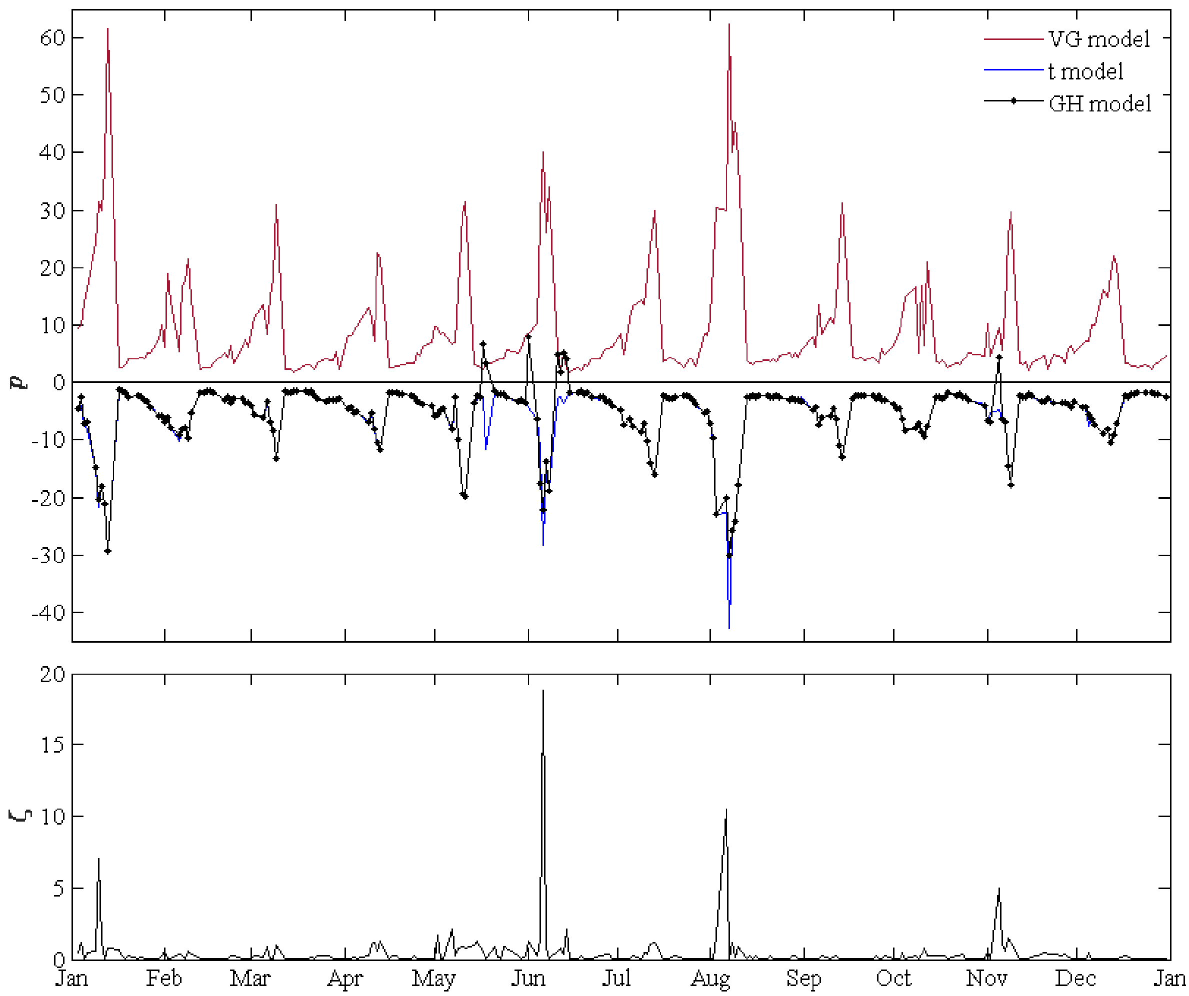
Figure 2: Daily estimates of the risk-neutral volatility, σ, and skewness, θ, parameters, with the S&P 500 Index and VIX.



The data used are S&P 500 Index puts observed in 2012. The parameter estimates depicted are those under the GH model. VIX is the volatility index, which measures the implied volatility of S&P 500 Index options.

Finally, for the GH model's second kurtosis parameter, ζ averages 0.417 while its maximum is 18.778. Outlying ζ values are observed in Figure 3 but they do not entail particularly worsened pricing errors. This may be explained by reference to Figure 1 of Subsection 2.2, which demonstrates that the GH option price is less sensitive to changes in ζ and p as the magnitudes of the parameters become larger. Regarding the special cases of the GH model, the minimum observed ζ is 3.547×10^{-14}. Coupled with the observation that $p > 0$ or $p < -1$ on all sample days, we can infer that on the days where $\zeta = 0$ (Figure 3), the GH model estimates a log-distribution which is either the VG ($p > 0$) or t ($p < -1$) special case. Indeed, the trajectory of the GH parameter p in the upper panel of Figure 3 appears to predominantly track the t model's estimated values for p, strongly suggesting that the t model may be a more plausible option pricing model than the VG model.

Figure 3: Daily estimates of risk-neutral kurtosis parameters, p and ζ.



The parameter ζ estimates are for the GH model. We may note that the special cases of the GH model are obtained as follows: hyperbolic at $p = 1$, normal reciprocal inverse Gaussian at $p = \frac{1}{2}$, normal inverse Gaussian at $p = -\frac{1}{2}$, reciprocal hyperbolic at $p = -1$, variance gamma at $\zeta = 0$ and $p > 0$, and the t model at $\zeta = 0$ and $p < -1$. Data used are S&P 500 Index puts observed in 2012.

We turn now to the first of the performance measures, in-sample pricing error. The daily-averaged root mean squared percentage error (RMSPE) for the GH model is the lowest, at 8.38% from Table 3. On the other hand, the Black-Scholes model starkly underperforms with an almost doubled daily-averaged RMSPE of 16.69%. As foreshadowed by the GH model's ζ and p estimates (Figure 3), between the VG and t models, the t model is the superior performer with an average daily RMSPE of 8.38% compared to the VG model's 9.50% RMSPE. The model with the most variable in-sample performance is the Black-Scholes model with the standard deviation of its daily RMSPE equal to 4.49%, followed by the VG model (RMSPE standard deviation of 3.81%) and then the t and GH models, with standard deviations of RMSPE equal to 3.28% and 3.27% respectively. The t and GH models also attain the best in-sample pricing accuracy of all models with their lowest daily RMSPE of 3.51%. The minimum RMSPE is inferior under the VG model, equal to 4.42%, and worst under the Black-Scholes model, equal to 8.85%.

Table 4: In-sample pricing root mean squared percentage errors (RMSPE) and mean absolute percentage errors (MAPE).

	In-sample RMSPE (%)				**In-sample MAPE (%)**			
	Time-to-maturity				Time-to-maturity			
	Short	**Medium**	**Long**	**All**	**Short**	**Medium**	**Long**	**All**
Out-of-the-money								
Black-Scholes	33.03	15.26	12.73	19.57	25.78	12.02	10.12	14.23
VG model	15.72	11.31	12.71	12.64	10.98	8.55	10.01	9.39
t model	*10.45*	10.80	*12.19*	*11.10*	*8.06*	*8.04*	*9.46*	*8.40*
GH model	10.47	*10.78*	*12.19*	*11.10*	8.07	*8.04*	*9.46*	*8.40*
At-the-money								
Black-Scholes	25.31	16.34	9.55	17.15	23.37	15.29	8.38	15.06
VG model	12.83	6.66	5.80	8.07	9.99	5.42	4.75	6.14
t model	9.63	*6.03*	5.56	6.79	7.93	*5.03*	4.51	*5.46*
GH model	*9.61*	*6.03*	*5.55*	*6.78*	*7.91*	*5.03*	*4.50*	*5.46*
In-the-money								
Black-Scholes	7.78	12.29	12.47	11.34	6.57	11.77	12.20	10.51
VG model	6.66	9.37	10.09	8.88	5.83	9.07	9.77	8.35
t model	*6.03*	*8.69*	*9.49*	*8.24*	*5.30*	*8.36*	*9.20*	*7.71*
GH model	6.13	8.70	9.54	8.27	5.35	8.37	9.25	7.73
All								
Black-Scholes	26.60	15.26	11.38	17.48	20.99	13.30	9.56	13.98
VG model	13.19	9.35	9.89	10.38	9.56	7.37	7.63	7.89
t model	*9.39*	8.79	*9.47*	*9.08*	*7.46*	*6.88*	*7.21*	*7.08*
GH model	9.40	*8.78*	*9.47*	*9.08*	*7.46*	*6.88*	*7.21*	*7.08*

The smallest error measure within each group is italicised. Parameters are estimated using all options on a given day, regardless of their time-to-maturity and moneyness (60 data points per sample, on average, and 250 samples). Whereas pricing errors are classified by time-to-maturity and moneyness in accordance with the categories in Table 2. The errors are then averaged across the 250 testing days collectively, $n = 15,058$, reconciling them with the daily-averaged RMSPE values in Table 3.

While Table 3 offered a gauge of overall in-sample fit, in Table 4 we examine how each model performs for various types of options, classified in terms of moneyness and time-to-maturity in accordance with Table 2. Between the two models that have the overall best in-sample fit, the GH and t models, we note two discrepancies in their cross-sectional fit. For at-the-money (ATM) options, the GH model is more accurate than the t model, particularly for short-term, ATM options where the GH model's RMSPE is 9.61% compared to 9.63% for the t model. Whereas for in-the-money (ITM) options, the t model has the pricing advantage over the GH model, with an 8.24% RMSPE compared to the GH model's 8.27%. The superior performance of the t model for ITM put options, indicates that the t distribution provides an optimal fit of the left tail of the log-distribution. We also note that between the VG and the t models, the t model fits all categories of options better than the VG model.

Furthermore, the Black-Scholes model's fit is best for ITM options, whereas the VG, t and GH models' fits are best for ATM options. The VG, t and GH models achieve the greatest pricing improvement compared to the Black-Scholes model in fitting ATM options. They reduce the Black-Scholes RMSPE from 17.15% to 8.07% (the VG model), 6.79% (the t model) and 6.78% (the GH model). For varying time-to-maturity, the Black-Scholes and VG models' pricing of short-term options are noticeably worse than the models' pricing of medium and long-term options. Meanwhile, the GH and t models' pricing performances are relatively even across time-to-maturity. Under the VG, t and GH models, it is short-term options that benefit most compared to the Black-Scholes model. The Black-Scholes RMSPE of 26.60% reduces to 13.19% under the VG model, and 9.39% and 9.40% under the t and GH models respectively. Lastly, results for the mean absolute percentage pricing error (MAPE) are also computed. While, the average overall MAPE was 2.00% lower than the average RMSPE, cross-sectionally the different error measure does not alter which model is superior in any given category.

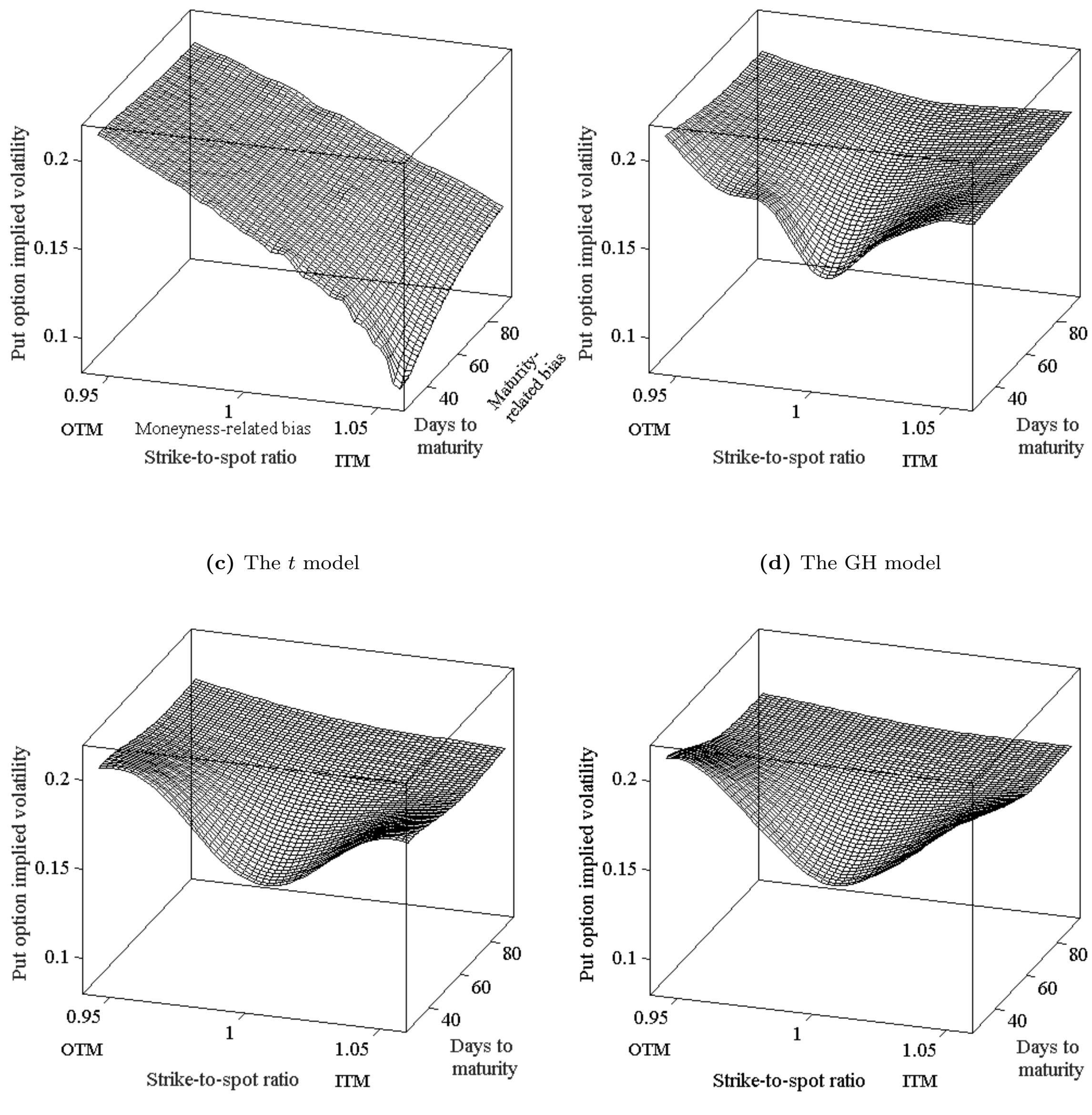
3.3 Orthogonality Test

A cross-sectional analysis of the pricing errors leads us to the second yardstick for assessing the models, which is also with respect to in-sample fit. A well specified model should not only achieve a minimal pricing error but should also return pricing errors that are independent of moneyness, time-to-maturity and interest rates (Rubinstein, 1985). From the Black-Scholes formula, which has a single parameter, there is a one-to-one mapping between the option price the option's implied volatility. As a result, the orthogonality of pricing errors can be ascertained via the orthogonality of option prices' implied volatilities (IV) (Eberlein, Keller, and Prause, 1998). In this section, we embark on this alternative and equivalent IV approach.

In order to calculate IV under the models (other than Black-Scholes), we solve for σ after equating the Black-Scholes option price to the estimated option prices under the VG, t and GH models. For Black-Scholes, IV is computed by equating the Black-Scholes option price to the observed price, equivalent to the Black-Scholes model's σ for an individual option. As a preliminary and non-comprehensive matter, Figure 4 shows the IV surfaces for an arbitrary sample day, Thursday, 17 May 2012. A validly specified model with orthogonal IV to moneyness and time-to-maturity should manifest a horizontal IV surface.

Subfigure 4(a) shows that the Black-Scholes IV surface is downward sloping, resembling a 'smirk' (Pan, 2002). On the other hand, the VG, t and GH models' IV surfaces are more horizontal. As one indication, the IV range under the Black-Scholes model of 8.6% to 21.0% reduces to a range of 14.0% to 21.0% under the VG model (Subfigure 4(b)), 14.8% to 20.4% under the t model (Subfigure 4(c)), and 14.9% to 20.9% under the GH model (Subfigure 4(d)). Inspecting specific cross-sections of the IV surfaces, all alternative models achieve corrections (slope flattening) of the Black-Scholes model's maturity-related bias (Bakshi, Cao, and Chen, 1997) for ITM puts and moneyness-related bias for longer-dated options. All three models' IV surfaces exhibit a quadratic form for short-term, ATM options.

Figure 4: Implied volatility surfaces.



(a) The Black-Scholes model

(b) The VG model

(c) The t model

(d) The GH model

An arbitrary day, Thursday, 17 May 2012, is sampled ($n = 63$) as a preview to the orthogonality test in Table 5. OTM and ITM refer to out-of-the-money and in-the-money.

To verify this misspecification diagnosis, we appeal to a linear regression model of the implied volatlities (Eberlein, Keller, and Prause, 1998). We include quadratic moneyness and quadratic time-to-maturity as regressors, in addition to their linear terms and the interest rate. Rather than observing a single day, we run the regression on the entire 250-day sample, containing 15,058 observations. The interpretation of the linear regression model's coefficient of determination is that a lower R^2 signifies greater orthogonality of IV. The regression model is as follows:

$$IV_i = b_0 + b_1 \left[\frac{K_i}{S_i}\right] + b_2 \left[\frac{K_i}{S_i}\right]^2 + b_3\,\tau_i + b_4\,\tau_i^2 + b_5\,r_i + e_i, \tag{9}$$

where for each observed put option price, IV_i is the implied volatility expressed as a decimal rather than as a percentage, τ_i is the time to maturity in years, K_i is the strike price, S_i and r_i are the spot S&P 500 Index and the risk-free interest rate (as a decimal) on the date the option price is observed, and e_i is the random error term.

Table 5: Orthogonality results.

Explanatory variable	**Black-Scholes**	**VG model**	***t* model**	**GH model**
Intercept	1.21	8.43	8.40	8.40
	(0.23)**	(0.17)**	(0.16)**	(0.16)**
Moneyness	-1.51	-16.54	-16.46	-16.46
	(0.47)**	(0.34)**	(0.33)**	(0.33)**
Moneyness^2	0.425	8.26	8.20	8.21
	(0.24)	(0.17)**	(0.16)**	(0.17)**
Time-to-maturity	0.024	-0.078	-0.077	-0.077
	(0.01)**	(0.01)**	(0.01)**	(0.01)**
Time-to-maturity^2	0.042	0.217	0.221	0.220
	(0.02)*	(0.02)**	(0.02)**	(0.02)**
Interest rate	-14.72	-15.19	-15.26	-15.34
	(0.45)**	(0.39)**	(0.38)**	(0.38)**
R^2	52.5%	24.1%	26.1%	26.0%
TSS	12.43	6.67	6.53	6.56
F-statistic	3329.2**	953.4**	1061.3**	1056.4**

S&P 500 Index put options from 2012 are used, n = 15,058, moneyness is the strike-to-spot price ratio, time-to-maturity is in years. Heteroskedastic standard errors are shown in parentheses, ** indicates statistical significance at a 1% level of significance, * indicates significance at a 5% level. The critical t-statistics are respectively $t_{(0.005,15052)} = \pm 2.58$ and $t_{(0.025,15052)} = \pm 1.96$. TSS is the total sum of squared errors. At a 1% level of significance, the critical $F_{(0.01,5,15052)}$-statistic is 3.02.

As presented in Table 5, for all models the F-statistics and coefficients for moneyness, time-to-maturity and the interest rate are statistically significant at a 1% level, using heteroskedasticity-consistent standard errors (White, 1980). As anticipated by Figure 4, the coefficients for the quadratic terms for moneyness and time-to-maturity are significant (at a 1% level) for only the VG, t and GH models. Also consistent with Figure 4, where the Black-Scholes' IV range reduces, the regression models' total sum of squared differences between observed IV and mean IV (TSS) similarly reduce from 12.43 (the Black-Scholes model) to 6.67 (the VG model), 6.53 (the t model) and 6.56 (the GH model). Not only however, does that the Black-Scholes model's regression's overall variation decrease, the proportion explained, as measured by R^2, also halves from 52.5% to 24.1% under the VG model, 26.1% under the t model, and

26.0% under the GH model. The lower R^2 and lower F-statistics highlight an increase in orthogonality, allowing us to deduce that all three models ameliorate the misspecification of the Black-Scholes model[16].

3.4 Out-of-Sample Pricing Performance

Finally, we turn to the out-of-sample pricing performance of the option pricing models. This third performance measure is motivated not only practically (to predict prices), but also statistically. Whilst in-sample pricing performance will always benefit from additional parameters, advantaging the GH model, the out-of-sample context can penalise overfitting (Bakshi, Cao, and Chen, 1997). In order to compute the out-of-sample pricing errors, parameters are estimated over a 5-day training period and then the option prices are predicted for the next day out of that period. Continuing with S&P 500 Index put options data for 2012, there are now 245 testing samples and a total of 14,794 pricing errors. On average each training sample has 301 data points.

Out-of-sample, the results in Table 6 show that the GH model emerges as the sole superior model overall in terms of both RMSPE (15.38%) and MAPE (11.04%). Unlike in the in-sample context, the GH model achieves greater accuracy than the t model. The t model attains a RMSPE of 16.33%, almost a full percentage point (0.95%) higher than the GH model, and a MAPE of 11.16% (0.12% higher than the GH model). Between the VG model and the t model, the t model is superior. The VG model obtains a higher RMSPE of 16.73% and a higher MAPE of 11.52%. All three models continue to prevail over the Black-Scholes model, which returns a RMSPE of 19.94% and a MAPE of 15.10%.

Cross-sectionally, the GH model is the best model for ATM, OTM, short-term and long-term options, outperforming the other models by a particularly wide margin for short-term options (a 2.97% margin) and OTM options (a 1.45% margin). It is the t model that fits the ITM options marginally better than the GH model. For the medium-term option prices, the VG model arises as the best predictor when MAPE is considered. We also remark that out-of-sample, the Black-Scholes model provides the best fit of OTM, long-term options. Lastly, focusing on the relative performance between the VG and t models, the t model offers a better out-of-sample fit for six of the nine moneyness and time-to-maturity combinations[17]. Compared to the in-sample setting, the out-of-sample pricing advantage of the t model over the VG model thus narrows.

[16]The VG model achieves a slightly lower R^2 and F-statistic than the GH and t models likely due to, as portrayed in Subfigure 4(b), a non-linear, non-quadratic dependency of IV on moneyness, which is not captured by the regression in Equation (9).

[17]We note that some pricing improvements are observed in the out-of-sample context compared to the in-sample context for ITM options (for example, ITM option prices as fit by the Black-Scholes and VG models). This can be explained by the relatively small representation of ITM options (16%) in the data compared to ATM (41%) and OTM (43%) options (see Table 2), such that the ITM options have a relatively weak influence over the in-sample parameter fitting.

Table 6: Out-of-sample pricing root mean squared percentage errors (RMSPE) and mean absolute percentage errors (MAPE).

	Out-of-sample RMSPE (%)				**Out-of-sample MAPE (%)**			
	Time-to-maturity				Time-to-maturity			
	Short	**Medium**	**Long**	**All**	**Short**	**Medium**	**Long**	**All**
Out-of-the-money								
Black-Scholes	36.35	19.36	*14.34*	23.35	26.95	15.38	*11.58*	17.04
VG model	33.77	17.48	15.74	21.87	22.12	13.99	13.13	15.60
t model	33.63	*17.27*	15.35	21.66	20.68	*13.86*	12.66	15.09
GH model	*29.31*	17.28	15.32	*20.21*	*19.52*	13.89	12.65	*14.85*
At-the-money								
Black-Scholes	27.98	17.61	10.06	18.61	23.11	15.38	8.49	15.06
VG model	20.44	*9.76*	*7.28*	12.10	14.52	*7.87*	*5.73*	8.59
t model	16.51	10.23	7.64	11.17	11.77	8.27	6.11	8.37
GH model	*15.18*	10.21	7.50	*10.77*	*11.56*	8.26	6.03	*8.31*
In-the-money								
Black-Scholes	7.14	12.10	11.59	10.98	5.92	11.26	10.93	9.87
VG model	6.45	9.35	*9.00*	8.66	5.50	8.61	*8.29*	7.78
t model	*5.64*	*9.08*	9.29	*8.39*	*4.79*	*8.27*	8.58	*7.45*
GH model	5.70	9.13	9.28	8.43	4.80	8.32	8.57	7.48
All								
Black-Scholes	29.88	17.60	12.30	19.94	21.70	14.68	10.13	15.10
VG model	26.01	13.55	11.96	16.73	16.31	*10.61*	9.30	11.52
t model	24.86	*13.54*	11.85	16.33	14.53	10.65	9.29	11.16
GH model	*21.89*	*13.54*	*11.79*	*15.38*	*13.93*	10.67	*9.25*	*11.04*

S&P 500 Index put options during 2012 are used. The smallest error measure within each group is italicised. The training sample is 5 days (on average containing 301 data points) and the testing sample is one day out of the training sample (on average containing 60 data points). There are 245 testing days. Parameters are estimated using all options. Whereas pricing errors are classified by time-to-maturity and moneyness in accordance with the categories in Table 2. The errors are then averaged across the 245 testing days collectively, $n = 14,794$.

4 Conclusion

Insight into the risk-neutral distribution of logarithmic stock returns is vital to the fitting and prediction of option prices. In this paper, we propose a flexible GH option pricing model, with four parameters all free to be estimated. We also present six three-parameter option pricing models, hosted by the flexible GH model: the VG, t, hyperbolic, reciprocal hyperbolic, normal inverse Gaussian and normal reciprocal inverse Gaussian option pricing models. In respect of the seven models' properties, the flexible GH model and its special cases generalise the Black-Scholes model by allowing the passage of economic time to depart from the deterministic process of physical time. As such, our class of flexible GH models are time-subordinated models, which can cope with yet another facet of the unpredictable financial market. In addition, the subordination to Brownian motion with drift means that the class of flexible

GH processes capture excess kurtosis and skewness.

Using S&P 500 Index options, we empirically compare the flexible GH option pricing model to the VG, t and Black-Scholes models. Our findings are three-fold. First, the flexible GH, VG and t models all reduce the Black-Scholes model's implied volatility smirk. Secondly, between the two three-parameter models, the weight of the empirical results supports the verdict that the t model is the more tenable model for pricing options. Remarkably, the t model's average in-sample fit is better than that of the VG model for all option types, a result which can be corroborated by the flexible GH model's parameter estimates. Out-of-sample, the t model also accomplishes a lower pricing error than the VG model for the majority of strike and maturity combinations. Ultimately however, we find that the assumption of generalised hyperbolically distributed log-returns has the greatest merit even in the out-of-sample context. With all four models considered, our flexible GH option pricing model attains the least absolute and squared out-of-sample pricing errors. Hence, a practitioner may prefer to use our flexible GH model to predict the prices of S&P 500 Index options over the VG, t and Black-Scholes models.

In sum, having reparameterised the GH option pricing model into a tractable form and validated it empirically, this paper sheds additional light on the distribution underlying option prices. Our flexible GH option pricing model however is a static model. Future work to improve the prediction of option prices may explore dynamic extensions to this paper's flexible GH model. Such an undertaking may begin with an empirical appraisal of Finlay and Seneta's GH option pricing model (2012), which represents one approach to incorporating time-dependence.

Appendix A

Subordinated Option Pricing Model Derivation

In this appendix, we derive the result in Equation (3). The price of a European call option at time t, with time-to-maturity, $\tau = T - t$, and strike price, K, is given by

$$\begin{aligned} C_t &= \mathrm{E}^Q[e^{-r\tau}(S_T - K)_+ | F_t] \\ &= S_t e^{-q\tau}\Pi_1 - K e^{-r\tau}\Pi_2, \end{aligned} \tag{A.1}$$

where $\mathrm{E}^Q[\cdot]$ denotes the expectation taken under the unique risk-neutral probability measure, Q, and Π_1 and Π_2 denote risk-neutral probabilities. From Bakshi and Madan (2000), Π_1 and Π_2 can be expressed in terms of the characteristic function of the logarithmic stock price, $\phi_{\log S_t}(u)$, through

$$\begin{aligned}\Pi_1 &= \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left(\frac{e^{-iw\log(K)}\phi_{\log S_t}(w-i)}{iw\phi_{\log S_t}(-i)}\right)dw \\ \Pi_2 &= \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left(\frac{e^{-iw\log(K)}\phi_{\log S_t}(w)}{iw}\right)dw.\end{aligned} \tag{A.2}$$

The price of a European put option is deduced by put-call parity as

$$P_t = Ke^{-r\tau}(1-\Pi_2) - S_t e^{-q\tau}(1-\Pi_1). \tag{A.3}$$

The characteristic function of the logarithmic stock price at time t, with S_t defined in Equation (2), Section 1 is given by

$$\begin{aligned}\phi_{\log S_t}(u) &= \mathrm{E}^Q[\exp\{iu\log(S_t)\}|F_0] \\ &= \mathrm{E}^Q[S_t^{iu}|F_0] \\ &= \mathrm{E}^Q\left[S_0\exp\left\{iu\left[(r-q)t + X_t + \omega t\right]\right\}\right] \\ &= S_0\exp\{iu(r-q+\omega)t\}\phi_{X_t}(u) \\ &= S_0\exp\{iu(r-q+\omega)t\}\phi_{X_1}(u)^t,\end{aligned} \tag{A.4}$$

since for Lévy processes such as X_t (defined in Equation (1), Section 1), $\phi_{X_t}(u) = \phi_{X_1}(u)^t$ (Barndorff-Nielsen, Mikosch, and Resnick, 2001).

Assuming an arbitrage-free and complete market, the discounted stock price process $S_t e^{-(r-q)t}$ is a martingale under Q (Harrison and Pliska, 1981). For the martingale to hold, the drift is adjusted with ω, such that for any positive t,

$$S_0 = \mathrm{E}^Q[S_t e^{-(r-q)t}|F_0]. \tag{A.5}$$

It follows from Equation (2) that

$$\begin{aligned}1 &= \mathrm{E}^Q[e^{X_t+\omega t}|F_0] \\ \omega &= -\frac{1}{t}\ln\phi_{X_t}(-i) \\ &= -\ln\phi_{X_1}(-i).\end{aligned} \tag{A.6}$$

Lastly, from the definition of X_t in Equation (1), the characteristic function of X_1 in Equation (A.4) can further be expressed in terms of the characteristic function of g_1 as derived below:

$$\begin{aligned}
\phi_{X_1}(u) &= \mathrm{E}^{Q}\left[\exp\left\{iu\left(\theta g_1+\sigma W\left(g_1\right)\right)\right\}\right] \\
&= \mathrm{E}_{g_1}\left\{\mathrm{E}^{Q|g_1}\left[\exp\left\{iu\left(\theta g_1+\sigma W\left(g_1\right)\right)\right\} \mid g_1\right]\right\} \\
&= \mathrm{E}_{g_1}\left\{\exp\left\{iu\theta g_1\right\}\mathrm{E}^{Q|g_1}\left[\exp\left\{iu\sigma W\left(g_1\right)\right\} \mid g_1\right]\right\} \\
&= \mathrm{E}_{g_1}\left[\exp\left\{iu\theta g_1\right\}\exp\left\{-\frac{1}{2}\sigma^2u^2g_1\right\}\right] \\
&= \mathrm{E}_{g_1}\left[\exp\left\{\left(iu\theta-\frac{1}{2}\sigma^2u^2\right)g_1\right\}\right] \\
&= \phi_{g_1}\left(u\theta+\frac{i\sigma^2u^2}{2}\right),
\end{aligned} \tag{A.7}$$

where the second equality follows from the law of iterated expectations. The outer expectation, denoted by $\mathrm{E}_{g_1}[\cdot]$, is taken with respect to the distribution of g_1. $\mathrm{E}^{Q|g_1}[\cdot]$ is the expectation taken under the risk-neutral measure, given g_1.

Appendix B

Derivations for the Flexible GH, VG and t Option Pricing Models

In this appendix, we derive the characteristic function of the logarithmic stock returns for the flexible GH, VG and t option pricing models presented in Equations (6), (7) and (8). The derivations use the characteristic functions of the generalised inverse Gaussian (GIG), gamma and reciprocal gamma distributions, respectively.

B.1 The flexible GH Model

From Jørgensen (1982), the characteristic function of the GIG distribution is given by

$$\begin{aligned}
\phi_{g_1}(u) &= \left(\frac{\gamma^2}{\gamma^2-2iu}\right)^{\frac{p}{2}}\frac{K_p\left(\sqrt{\delta^2(\gamma^2-2iu)}\right)}{K_p(\delta\gamma)} \\
&= \left(1-\frac{2}{\gamma^2}iu\right)^{-\frac{p}{2}}\frac{K_p\left(\zeta\sqrt{1-\frac{2}{\gamma^2}iu}\right)}{K_p(\zeta)},
\end{aligned} \tag{A.8}$$

where $\zeta=\delta\gamma$, γ^2 is given by Equation (5) and $K_h(\cdot)$ is the modified Bessel function of the third kind with index h. Using the result in Equation (A.7), we obtain

$$\begin{aligned}
\phi_{X_1}(u) &= \phi_{g_1}\left(u\theta+\frac{i\sigma^2u^2}{2}\right) \\
&= \left(1-\frac{2}{\gamma^2}\left(iu\theta-\frac{1}{2}\sigma^2u^2\right)\right)^{-\frac{p}{2}}\frac{K_p\left(\zeta\sqrt{1-\frac{2}{\gamma^2}\left(iu\theta-\frac{1}{2}\sigma^2u^2\right)}\right)}{K_p(\zeta)}.
\end{aligned} \tag{A.9}$$

It follows from Equation (A.4) that

$$\begin{aligned}\phi_{\log S_t}(u) &= S_0 \exp\{iu(r-q+\omega)t\}\phi_{X_1}(u)^t \\ &= S_0 \exp\{iu(r-q+\omega)t\}\left[1-\frac{2}{\gamma^2}\left(iu\theta-\frac{1}{2}\sigma^2u^2\right)\right]^{-\frac{pt}{2}} \times \\ &\left[\frac{K_p\left(\zeta\sqrt{1-\frac{2}{\gamma^2}\left(iu\theta-\frac{1}{2}\sigma^2u^2\right)}\right)}{K_p(\zeta)}\right]^t,\end{aligned} \tag{A.10}$$

where the drift adjustment can be computed using Equation (A.6) as

$$\begin{aligned}\omega &= -\ln\phi_{X_1}(-i) \\ &= -\frac{p}{2}\ln\left[\left(1-\frac{2}{\gamma^2}\left(\theta+\frac{1}{2}\sigma^2\right)\right)\right]+\ln\left[\frac{K_p\left(\zeta\sqrt{1-\frac{2}{\gamma^2}\left(\theta+\frac{1}{2}\sigma^2\right)}\right)}{K_p(\zeta)}\right],\end{aligned} \tag{A.11}$$

for $\theta < \left(\frac{\gamma^2}{2}-\frac{\sigma^2}{2}\right)$.

B.2 The VG Model

The characteristic function of the gamma distribution is given by $\phi_{g_1}(u) = (1-iu\nu)^{-\frac{1}{\nu}}$ such that

$$\phi_{X_1}(u) = \left[1-\nu\left(iu\theta-\frac{1}{2}\sigma^2u^2\right)\right]^{-\frac{1}{\nu}}. \tag{A.12}$$

It follows then that the VG option price can be computed using

$$\phi_{\log S_t}(u) = S_0\exp\{iu(r-q+\omega)t\}\left[1-\nu\left(iu\theta-\frac{1}{2}\sigma^2u^2\right)\right]^{-\frac{t}{\nu}}, \tag{A.13}$$

with drift adjustment, $\omega = \frac{1}{\nu}\ln\left[1-\nu\left(\theta+\frac{1}{2}\sigma^2\right)\right]$, for $\theta < \left(\frac{1}{\nu}-\frac{\sigma^2}{2}\right)$.

B.3 The t Model

The characteristic function of the reciprocal gamma distribution is given by

$$\begin{aligned}\phi_{g_1}(u) &= \frac{2(-\beta iu)^{\alpha/2}}{\Gamma(\alpha)}K_\alpha\left[\sqrt{-4\beta iu}\right] \\ &= \frac{2(\frac{1}{\nu}-1)^{\frac{1}{2\nu}}}{\Gamma(\frac{1}{\nu})}(-iu)^{\frac{1}{2\nu}}K_{\frac{1}{\nu}}\left[\sqrt{-4\left(\frac{1}{\nu}-1\right)iu}\right],\end{aligned} \tag{A.14}$$

where $\alpha = \frac{1}{\nu}$ and $\beta = \frac{1}{\nu}-1$ for $0 < \nu < 1$. $K_h(\cdot)$ is the modified Bessel function of the third kind with index h. Using the same procedure as for the flexible GH and VG derivations, the characteristic function of X_1 is given by

$$\phi_{X_1}(u) = \frac{2\left(\frac{1}{\nu}-1\right)^{\frac{1}{2\nu}}}{\Gamma(\frac{1}{\nu})}\left[-\left(iu\theta-\frac{1}{2}\sigma^2u^2\right)\right]^{\frac{1}{2\nu}} K_{\frac{1}{\nu}}\left[\sqrt{-4\left(\frac{1}{\nu}-1\right)\left(iu\theta-\frac{1}{2}\sigma^2u^2\right)}\right]. \tag{A.15}$$

The characteristic function of $\log S_t$ will be

$$\begin{aligned}&\phi_{\log S_t}(u) = S_0\exp\{iu(r-q+\omega)t\}\times\\ &\left[\frac{2\left(\frac{1}{\nu}-1\right)^{\frac{1}{2\nu}}}{\Gamma(\frac{1}{\nu})}\left[-\left(iu\theta-\frac{1}{2}\sigma^2u^2\right)\right]^{\frac{1}{2\nu}} K_{\frac{1}{\nu}}\left(\sqrt{-4\left(\frac{1}{\nu}-1\right)\left(iu\theta-\frac{1}{2}\sigma^2u^2\right)}\right)\right]^t,\end{aligned} \tag{A.16}$$

and the drift adjustment,

$$\omega = -\ln\left[\frac{2\left(\frac{1}{\nu}-1\right)^{\frac{1}{2\nu}}}{\Gamma(\frac{1}{\nu})}\left[-\left(\theta+\frac{1}{2}\sigma^2\right)\right]^{\frac{1}{2\nu}} K_{\frac{1}{\nu}}\left(\sqrt{-4\left(\frac{1}{\nu}-1\right)\left(\theta+\frac{1}{2}\sigma^2\right)}\right)\right], \tag{A.17}$$

for $\theta < -\frac{\sigma^2}{2}$.

Appendix C

The VG and t Limiting Cases of the Flexible GH Model

(i) Suppose $p = \frac{1}{\nu} > 0$, $\delta \to 0$ and $\gamma = \sqrt{\frac{2}{\nu}}$. Then, the flexible GH model reduces to the VG model.

(ii) Suppose $p = -\frac{1}{\nu} < -1$, $\gamma \to 0$ and $\delta = \sqrt{2(\frac{1}{\nu}-1)}$. Then, the flexible GH model reduces to the t model.

Proof. Either $\delta \to 0$ or $\gamma \to 0$ implies that $\zeta \to 0$. Making use of the properties of the modified Bessel function of the third kind with index h, $K_h(\cdot)$, that

$$K_h(y) \sim \Gamma(|h|)2^{|h|-1}y^{-|h|} \qquad \text{for } y \downarrow 0,$$

and that $K_h(y) = K_{-h}(y)$, the characteristic function of X_1 under the flexible GH model (Equation (A.9)) can be further simplified as follows.

Under the assumptions in (i), we have

$$\begin{aligned}
\phi_{X_1}(u) &= \left(1-\nu\left(iu\theta-\frac{1}{2}\sigma^2u^2\right)\right)^{-\frac{1}{2\nu}} \lim_{\delta\to 0}\frac{\Gamma(\frac{1}{\nu})2^{\frac{1}{\nu}-1}\left(\delta\gamma\sqrt{1-\nu\left(iu\theta-\frac{1}{2}\sigma^2u^2\right)}\right)^{-\frac{1}{\nu}}}{\Gamma(\frac{1}{\nu})2^{\frac{1}{\nu}-1}\left(\delta\gamma\right)^{-\frac{1}{\nu}}} \\
&= \left(1-\nu\left(iu\theta-\frac{1}{2}\sigma^2u^2\right)\right)^{-\frac{1}{2\nu}}\left(1-\nu\left(iu\theta-\frac{1}{2}\sigma^2u^2\right)\right)^{-\frac{1}{2\nu}} \\
&= \left(1-\nu\left(iu\theta-\frac{1}{2}\sigma^2u^2\right)\right)^{-\frac{1}{\nu}},
\end{aligned}$$

which is the corresponding characteristic function under the VG model, given in Equation (A.12).

Under the assumptions in (ii), we have

$$\begin{aligned}
\phi_{X_1}(u) &= \lim_{\gamma\to 0}\left(1-\frac{2}{\gamma^2}\left(iu\theta-\frac{1}{2}\sigma^2u^2\right)\right)^{\frac{1}{2\nu}}\frac{K_{\frac{1}{\nu}}\left(\delta\sqrt{\gamma^2-2\left(iu\theta-\frac{1}{2}\sigma^2u^2\right)}\right)}{K_{\frac{1}{\nu}}(\delta\gamma)} \\
&= \lim_{\gamma\to 0}\left(1-\frac{2}{\gamma^2}\left(iu\theta-\frac{1}{2}\sigma^2u^2\right)\right)^{\frac{1}{2\nu}}\frac{K_{\frac{1}{\nu}}\left(\delta\sqrt{\gamma^2-2\left(iu\theta-\frac{1}{2}\sigma^2u^2\right)}\right)}{\Gamma(\frac{1}{\nu})2^{\frac{1}{\nu}-1}\left(\delta\gamma\right)^{-\frac{1}{\nu}}} \\
&= \lim_{\gamma\to 0}\frac{\delta^{\frac{1}{\nu}}}{\Gamma(\frac{1}{\nu})2^{\frac{1}{\nu}-1}}\left(\gamma^2-2\left(iu\theta-\frac{1}{2}\sigma^2u^2\right)\right)^{\frac{1}{2\nu}}K_{\frac{1}{\nu}}\left(\delta\sqrt{\gamma^2-2\left(iu\theta-\frac{1}{2}\sigma^2u^2\right)}\right) \\
&= \frac{\delta^{\frac{1}{\nu}}}{\Gamma(\frac{1}{\nu})2^{\frac{1}{\nu}-1}}\left(-2\left(iu\theta-\frac{1}{2}\sigma^2u^2\right)\right)^{\frac{1}{2\nu}}K_{\frac{1}{\nu}}\left(\delta\sqrt{-2\left(iu\theta-\frac{1}{2}\sigma^2u^2\right)}\right) \\
&= \frac{2^{\frac{1}{2\nu}}\left(\frac{1}{\nu}-1\right)^{\frac{1}{2\nu}}}{\Gamma(\frac{1}{\nu})2^{\frac{1}{\nu}-1}}2^{\frac{1}{2\nu}}\left[-\left(iu\theta-\frac{1}{2}\sigma^2u^2\right)\right]^{\frac{1}{2\nu}}K_{\frac{1}{\nu}}\left[\sqrt{-4\left(\frac{1}{\nu}-1\right)\left(iu\theta-\frac{1}{2}\sigma^2u^2\right)}\right] \\
&= \frac{2\left(\frac{1}{\nu}-1\right)^{\frac{1}{2\nu}}}{\Gamma(\frac{1}{\nu})}\left[-\left(iu\theta-\frac{1}{2}\sigma^2u^2\right)\right]^{\frac{1}{2\nu}}K_{\frac{1}{\nu}}\left[\sqrt{-4\left(\frac{1}{\nu}-1\right)\left(iu\theta-\frac{1}{2}\sigma^2u^2\right)}\right],
\end{aligned}$$

which is the corresponding characteristic function under the t model, given in Equation (A.15). ■

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