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Abstract

We examine innovation as a market-entry timing game with complete information and observable actions. We characterize all pure-strategy subgame perfect equilibria for the two-player symmetric model allowing both the leader's and the followers' payoff functions to be multi-peaked, non-monotonic and discontinuous. We provide sufficient conditions for when the equilibria can be Pareto ranked and when the equilibrium is unique. Economic applications discussed include process and product innovation and the timing of the sale of an asset.

Key words: timing games, entry, leader, follower, process innovation, product innovation.

JEL classifications: C72, L13, O31, O33.

1 Introduction

The availability of new products and processes underlies economic development and improvements in welfare (Romer, 1994). But new technology does not automatically equate to innovation in the market place. Rather, any innovation – be it market entry with a new product or adoption of a new production process – must be deliberately implemented as part of a firm's profit-maximizing strategy. In this paper, following Fudenberg and Tirole (1985), Dutta et al. (1995) and Hoppe and Lehmann-Grube (2005), we study an innovation-timing game in which two competing firms consider the optimal time to enter a market. We extend these existing models by generalizing both the leader's and the follower's profit functions.

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When considering the optimal time to innovate, a monopolist must weigh up several different factors. If it moves early, it enjoys the benefits of innovation sooner and for longer. By waiting, a monopolist might be able to implement a better – and possibly more profitable – innovation. This is the case when the potential quality of the good improves over time, as in a product-innovation model. Similarly, waiting could allow the monopolist to implement the same innovation but at lower cost (a process-innovation model). In either situation, there is a benefit from delaying entry into the market. A monopolist will weigh up the benefits of early-versus-late entry and choose an entry time so as to maximize its net present value of innovating.[1]

While similar tradeoffs exist in an oligopoly, the strategic interaction between firms also needs to be taken into account. For example, consider two firms contemplating the best time to release a new phone in the sort of dilemma Apple and Samsung face when launching a new handset. There could be a first-mover advantage in this market; a leader could develop a loyal customer base and a network of related products, helping secure its market dominance in the long run. But waiting and entering second could also be advantageous, allowing a firm to launch a better phone, with more features, memory, and so on, possibly even at lower cost.[2] Each firm will weigh up the relative advantage of early rather than late entry, taking into account the strategy of their rival.

Fudenberg and Tirole (1985) analyze the adoption of new technology by two rivals, neither of whom are able to pre-commit to their strategy.[3] They develop a method of solving the continuous-time game using subgame perfection that we adopt here. An important insight in their paper is that with a first-mover advantage there is a preemption equilibrium in which all the rents from entering the market as the leader are dissipated by 'excessively' early entry (that is, entry occurs at a time much earlier than a monopolist would choose). Furthermore, in this equilibrium: rents for the leader and follower are equalized; and entry times display diffusion, à la Reinganum (1981a), in which one firm adopts early while the follower waits and enters the market later. In addition, Fudenberg and Tirole (1985) show there can be a continuum of joint-adoption equilibria that also involve rent equalization for the two firms.

In solving their model, Fudenberg and Tirole (1985) assume profits at any point in time depend on whether a firm and its rival have entered, and not on how long either firm has been active in the market. Using a vertical product-differentiation model, Dutta et al. (1995) extends Fudenberg and Tirole (1985) to study entry when the potential quality of the product improves over time. In contrast to Fudenberg and Tirole (1985), Dutta et al. (1995) assume that profits depend only on the difference of entry times. They show that in their model there still can be: a preemption equilibrium, in which rents are dissipated through excessively early entry; or an equilibrium in which no monopoly profit is dissipated but the follower makes higher return than the market leader.

[1] Given it does not, in general, capture all of the surplus from innovation, a monopolist typically will not innovate at the socially optimal time; see Tirole (1988, Chapter 10).

[2] See Tellis and Golder (1996) for a study on second-mover advantages in a range of markets.

[3] Fudenberg and Tirole (1985) also consider a general setup with n players.

Fudenberg and Tirole (1985) effectively assume that the follower's entry time is exogenously determined when the leader enters before a given time, after which the leader's and the follower's time of entry coincide. This implies that while Fudenberg and Tirole (1985) allow for multiple peaks in the leader's payoff, all but the first peak must perfectly coincide with the follower's payoff. Hoppe and Lehmann-Grube (2005) extend the analysis of Fudenberg and Tirole (1985) by allowing the leader's payoff curve to have multiple peaks (local maxima). Hoppe and Lehmann-Grube (2005), on the other hand, require that the follower's payoff is non-increasing in the leader's entry time. They solve for both the preemption and the second-mover advantage games.

In this paper we generalize these existing models in several dimensions. First, we solve for the pure-strategy subgame perfect equilibria when payoffs for both firms can be non-monontonic or multi-peaked. In this way our framework allows us to solve a broader range of economic problems than was previously possible. For example, in Section 3.1 we show that a process-innovation or a product-innovation model, augmented with an experience good or some switching cost, can generate a non-monotonic payoff for both the leader and the follower. The same point can be made for the profit derived from an asset; the revenue generated can vary non-monotonically depending on the time of sale. Unlike existing methods, the solution algorithm developed here is able to allow for any possible continuous payoff structure.

Second, our model is sufficiently general to accommodate discontinuities in payoffs. Discontinuities arise in a variety of situations; at some point in time (in terms of the leader's entry time) a firm in a related complementary or substitute market could enter or decide to exit.[4] This decision could create a discontinuity in either the leader's or the followers' payoff (or both). For instance, in the phone-handset example above, developers of apps could enter or exit, affecting discontinuously the payoff to either the leader or the follower. Similarly, the product choices of firms selling substitute products, such as tablets, could also disrupt the phone-handset sellers, generating discontinuities.

Some of the key results in the paper are as follows. In characterizing all pure-strategy subgame perfect equilibria we find that there can be multiple equilibria. First, there could be a set of equilibria that exhibit rent equalization. The leader's entry times in these equilibria occur at times when the leader and follower payoff curves intersect and the leader's payoff is at a historic maximum for the game up until that time; they are similar to the joint-adoption equilibria in Fudenberg and Tirole (1985). Second, equilibria can exist with the leader entering at points of discontinuity, for example if the leader receives a higher payoff than the follower at this time, and that the expected payoff in equilibrium is higher than the payoff from entering as a leader at any earlier time. An example of this is immediate entry at

[4] As noted by Bobtcheff and Mariotti (2012), many factors that affect an entrant's profitability are exogenous, outside of the control of the firms themselves. These events could see a discontinuous jump in the payoffs of the leader and/or the followers. Fudenberg and Tirole (1985, Section 5) also discuss how three (or more) firms can generate a discontinuity in payoffs for the remaining firms in a timing game similar to the one we study here.

the very start of the game when both firms prefer to be first into the market. Third, there can be equilibria with asymmetric payoffs, like the second-mover advantage equilibrium of Hoppe and Lehmann-Grube (2005) and the maturation equilibrium of Dutta et al. (1995). Finally, when there are multiple equilibria we provide sufficient conditions to ensure that these equilibria can be Pareto ranked. We also outline sufficient conditions for when the subgame perfect equilibrium is unique.

This paper draws on an extensive literature on innovation timing games.[5] Our analysis of an irreversible investment decision with complete information and observable actions (closed-loop equilibria) follows Fudenberg and Tirole (1985), Dutta et al. (1995) and Hoppe and Lehmann-Grube (2005). This framework has been used to study a variety of applications. For example, Argenziano and Schmidt-Dengler (2012, 2013, 2014) adopt a variant of Fudenberg and Tirole (1985) to examine the order of market entry, clustering and delay. They show that with many firms the most efficient firm need not be the first to enter the market and that delays are non-monotonic with the number of firms. In addition, they suggest a new justification for clustering of entry.[6]

An alternative approach to study innovation is to assume players' actions are unobservable as in Reinganum (1981a) and Reinganum (1981b). In her models, unobservable actions are equivalent to each firm being able to pre-commit to its strategy at the start of the game. Reinganum shows that in the (open-loop) equilibria there will be diffusion in the sense that firms adopt the technology at different dates, even though all firms are ex ante identical. Park and Smith (2008) develop an innovation game with unobservable actions that permits any firm (in terms of the order of entry) to receive the highest payoff. This allows for a war-of-attrition, with higher payoffs for late movers, a pre-emption game with higher payoffs for early movers, and a combination of both. They solve for the (open-loop) mixed-strategy equilibria.[7] In our model firms use feedback rules to determine their strategy at any particular point in time; this means that they are unable to commit to their strategy at the beginning of the game.

Finally, several other authors consider innovation when there is asymmetric information. For example, Bobtcheff and Mariotti (2012), Hendricks (1992) and Hopenhayn and Squintani (2011) assume that a firm's capability to innovate is private information. In these models, delay allows a firm to get better information about the potential innovation (its costs, value, and so on), but waiting runs the risk that a rival will innovate first, capturing the lion's share of the returns.

[5]See Hoppe (2002) or Van Long (2010, Chapter 5) for a survey of the literature. Further, Fudenberg and Tirole (1991) consider innovation when the firms make one irreversible decision (to enter) in a simple timing-game framework (see Sections 4.5 and 4.12).

[6]Timing games have been studied in a number of other contexts. Katz and Shapiro (1987) analyze an innovation game with heterogenous firms when there is licencing (by the leader) and imitation (by the follower). Dutta and Rustichini (1993) consider a stochastic timing game with continuous payoffs. Gale (1995) shows that inefficient delays can occur when n players make a one-off investment decision in a dynamic coordination game.

[7]They also briefly consider observable actions and show that there are multiple equilibria.

2 The model

Assume two firms ($i = 1, 2$) are in a continuous-time stopping game starting at $t = 0$ until some terminating time $T \in (0, \infty]$. Firm i's decision to stop (that is, 'enter' the market) at $t_i \geq 0$ can only be made once, and this decision is irreversible and observable immediately by the other firm. The game ends when both players have stopped. Firm i's payoff depends on the stopping times of both firms: $\pi_i(t_1, t_2)$. If the game ends with the two players stopping at different times, assume that the payoffs of the leader and the follower are $L(t_1, t_2) = \pi_i(t_1, t_2)$ and $F(t_1, t_2) = \pi_j(t_1, t_2)$, respectively, $t_i < t_j$ where $i, j = 1, 2$ and $i \neq j$.

We make the following standard assumptions.

Assumption 1. *Time is continuous in that it is 'discrete but with a grid that is infinitely fine'.*

Assumption 2. *Firms always choose to stop earlier rather than later in payoff-equivalent situations.*

Assumption 3. *If more than one firm chooses to stop (enter) at exactly the same time, one of these firms is selected to stop (each with probability $\frac{1}{2}$ ex ante); the other firm is then able to reconsider its decision to stop at this time.*

Equivalent assumptions are adopted in the literature. For example, Assumption 1 replicates A1 of Hoppe and Lehmann-Grube (2005). It invokes Simon and Stinchcombe (1989) who show that under certain conditions a continuous-time strategy profile is the limit of a discrete-time game with increasingly fine time grids.[8] Assumption 2, which is very similar to A3 in Hoppe and Lehmann-Grube (2005), allows us to focus on just one (payoff-equivalent) equilibrium in the case of indifference between early and late entry.[9] This simplifies our analysis so as to focus on the timing of entry rather than on issues of equilibrium selection.

Assumption 3 – part of A3 in Hoppe and Lehmann-Grube (2005) and Assumption 5 in Dutta et al. (1995) – avoids potential coordination failures involving simultaneous entry. Given its importance, further discussion of the intuition underlying this assumption is worthwhile. This assumption can be justified in several ways. In some situations, as a practical matter, if two firms try to enter the market at the same time there might be some capacity constraint or institutional requirement that prevents joint entry – consequently, only one firm becomes the leader and the other firm is relegated to the role of second entrant. For example, in a particular market there could be a bureaucratic rule that requires that the leadership role be allocated to the firm that has the first email registered in a designated inbox. Even if both firms simultaneously send their messages, only one email can arrive first. As a consequence,

[8]See Hoppe and Lehmann-Grube (2005), footnote 4 for a further discussion.

[9]Hoppe and Lehmann-Grube (2005) assume that if the follower is indifferent between two alternative entry times, it chooses the earliest time. For consistency, we extend this assumption to both firms.

with simultaneous moves, each firm has some probability of being the leader. In our model Assumption 3 gives either firm an equal chance of having its email received first.[10]

Following Fudenberg and Tirole (1985) we use subgame perfection as our equilibrium concept. A history h_t is defined as the knowledge of whether or not firm $i = 1, 2$ previously stopped at any time $\hat{t} < t$, and if so when. A strategy of firm i, denoted by $\sigma_i(h_t)$, indicates at each history h_t whether firm i stops at t ($\sigma_i(h_t) = 1$) or does not stop at t ($\sigma_i(h_t) = 0$) if it has not already done so. A strategy pair (σ_1, σ_2) maps every history to an outcome, which is a pair of stopping times (t_1, t_2). As usual, a strategy profile (σ_1^*, σ_2^*) constitutes a subgame perfect equilibrium (SPE) if the strategies are sequentially rational after every history.

As we will show, there is the possibility of multiple SPE in our game. When this is the case, sometimes it is possible to Pareto rank the equilibria and determine the superior subgame perfect equilibrium (SSPE) – that is, the SPE that Pareto-dominates all other SPE. Fudenberg and Tirole (1983) and Fudenberg and Tirole (1985) argue that this equilibrium would be a natural 'focal point' for firms in the game. Explicitly, we define the SSPE to be the equilibrium in which all firms receive a payoff at least as high as they could have received in any other SPE. In our model, we give sufficient conditions when it is possible to Pareto rank the SPE and determine the SSPE of the game. In addition, for convenience, we label the SPE that provides the leader with the highest possible payoff as the leader's preferred subgame perfect equilibrium (LSPE).

With two firms, the follower's entry is a single-firm decision problem, and its entry time will be its unique best response given the leader's choice. Without loss of generality assume that firm 1 is the leader whereas firm 2 is the follower, so that $t_1 \leq t_2$. From this, we can write the follower's entry time $t_2(t_1)$. Moreover, the payoffs to both firms can be written as composite functions of the leader's time of entry. With a slight abuse of notation $L(t_1) = L(t_1, t_2(t_1))$ and $F(t_1) = F(t_1, t_2(t_1))$ are the payoffs to the leader and the follower, respectively. Note here that with this representation we only need to specify the strategies when there has been no entry in the history of the game, because we assume that once one firm has entered, its rival will adopt its best response. This allows us, for ease of exposition, to refer from hereon to each firm's entry strategy as a function of time only, $\sigma_i(t)$.

Using this new terminology, let us now outline the next assumption.[11]

Assumption 4. *There exists a finite $t^{max} < T$, which is the earliest time at which $L(t)$ attains its global maximum. Specifically, $L(t^{max}) > L(\tau)\ \forall\ \tau < t^{max}$, and $L(t^{max}) \geq L(\tau)\ \forall\ \tau \geq t^{max}$.*

A similar assumption is adopted by others in the literature; it ensures that the

[10]Dutta et al. (1995) present a similar rationale for this assumption, suggesting there could be small random delays between when a decision is made and when a new technology is adopted that provide some probability that either firm will be first in the event of joint adoption.

[11]When there is no ambiguity, we refer to payoffs as a function of t rather than t_1.

leader stops in finite time. For example, this is equivalent to Assumption 3 in Dutta et al. (1995) and Assumption 2(ii) in Fudenberg and Tirole (1985).

Finally, our last assumption ensures that both firms innovate at $t_i \leq T \ \forall \ i$ as entering provides a higher payoff than its outside option of zero. This means that our analysis is not unnecessarily complicated by having to consider the case when one or both firms never enter the market.

Assumption 5. *Each firm's outside (non-entry) payoff is normalized to 0, and* $L(t) \geq 0$ *and* $F(t) \geq 0$.

This Assumption plays a similar role to Assumption 4 in Dutta et al. (1995) and Assumption 2(ii) in Fudenberg and Tirole (1985).

In summary, these five assumptions are standard in the market-entry timing game literature with complete information. Within this framework our model is the most general structure possible consistent with these previous papers.

3 Continuous payoffs

Let us first consider two symmetric firms with continuous payoff functions $L(t)$ and $F(t)$. While this setup is similar to Hoppe and Lehmann-Grube (2005), an important departure is that in our paper $F(t)$ can be non-monotonic.

Here, we develop a method to determine the leader's time of the entry in all SPE. To find the SPE in our timing game, we note that any equilibrium entry time t^* must satisfy two necessary conditions:

Condition 1. *No Preemption by the Leader (NPL):* $L(t^*) > L(\tau) \ \forall \ \tau < t^*$.

Condition 2. *No Preemption by the Follower (NPF):* $F(t^*) \geq L(t^*)$.

The *NPL* is required in any SPE, otherwise the leader will opt to enter earlier. Similarly, the *NPF* must hold in any SPE, otherwise the follower will have an incentive to preempt the leader and enter slightly earlier, as in Fudenberg and Tirole (1985). Even if these conditions hold it does not guarantee that an entry time will be part of an SPE, because the conditions only compare payoffs at a particular time relative to its historical values. There is nothing in these conditions involving a comparison with future potential payoffs, which is necessary when deriving an SPE.

To solve for the leader's entry time, let us eliminate all points that do not satisfy either of these conditions by constructing a set $A(t')$, defined as

$$A(t') = \{t \geq t' \ | \ F(t) \geq L(t) > L(\tau) \ \forall \ \tau \in [t', t)\}. \tag{1}$$

A point belongs to set $A(t')$ if it satisfies both *NPL* and *NPF*. Now we are in a position to consider the leader's preferred SPE (LSPE), as presented in the following lemma.

Lemma 1. *In the LSPE, which always exists, the first firm's stopping time is:*

$$t^* = \begin{cases} \arg\max_t A(0) & \textit{if} \quad A(0) \neq \emptyset, \\ 0 & \textit{otherwise.} \end{cases} \tag{2}$$

The strategies firms adopt in the LSPE are:

$$\sigma_1(t) = \begin{cases} 1 & \textit{if} \quad A(t) = \{t\}, \\ 0 & \textit{otherwise}; \end{cases} \tag{3}$$

$$\sigma_2(t) = \begin{cases} 1 & \textit{if} \quad L(t) \geq F(t) \ \& \ A(t) = \{t\}, \\ 0 & \textit{otherwise.} \end{cases} \tag{4}$$

Proof: See Appendix A.

Lemma 1 describes the solution for the SPE that provides the leader with its highest possible payoff, allowing for any continuous $L(t)$ and $F(t)$ payoff functions. The equilibrium strategy of firm 1 is to enter whenever there is no additional gain from delaying entry – this is represented here by the condition $A(t) = \{t\}$. On the other hand, the equilibrium strategy of firm 2 is to wait unless they are (weakly) better off being a leader at a given time t. This is represented by two conditions: $L(t) \geq F(t)$; and $A(t) = \{t\}$. The first condition means that they are (weakly) better off being a leader rather than a follower at a given time t, while the second condition means they prefer being a leader at t rather than at some later time.

Note that firms have different strategies to allow for asymmetries in equilibria. Given each firm is otherwise identical, to avoid coordination failures in which both firms enter at the same time we assume, for convenience, that firm 1 has a slightly weaker bargaining position in comparison with firm 2 so that it receives (or is willing to 'accept') the lower payoff available in this LSPE. With these somewhat 'predetermined' roles, firm 1 becomes the leader when both firms prefer to be the follower.[12]

In equilibrium, we observe the leader enter the market immediately when either $A(0) = \emptyset$ or $A(0) = \{0\}$. In the first case, $L(0) > F(0)$ and there is no benefit from waiting because $A(0)$ is empty. In the second case, $L(0) \leq F(0)$, but as $A(0) = \{0\}$ there is again no advantage in delaying entry in equilibrium. Alternatively, entry by the leader occurs after a delay when $t^* = \arg\max_t A(0) > 0$. In this case, there is an advantage of waiting until t^*. Note, the LSPE is always the SPE with the latest possible entry time for the leader.

Having outlined the LSPE, we are able to describe all the equilibria of the game (which also include the LSPE). First let us consider the equilibria that occur when the leader and follower curves coincide or intersect; note that there are similar joint-adoption equilibria in Fudenberg and Tirole (1985). To do this we introduce the following condition:

[12] Note an equivalent assumption is made in the previous entry-game literature in order to avoid the coordination issues; see for example, Fudenberg and Tirole (1985), Dutta et al. (1995) and Hoppe and Lehmann-Grube (2005).

Condition 3. *Rent equalization (RE):* $F(t^*) = L(t^*)$.

To solve for SPE with rent equalization we introduce set $B \subset A(0)$ that contains all points satisfying condition *NPL*, *NPF* and *RE*, that is:

$$B = \{\ t \ \mid\ F(t) = L(t) > L(\tau)\ \ \forall\ \ \tau \in [0, t)\}. \tag{5}$$

Using this set we can present a lemma that describes all SPE of the game in which the RE condition holds. The players' strategies in these SPE are also outlined.

Lemma 2. *For any* $t^* \in B$ *there is a corresponding SPE with rent equalization in which both firms enter at* t^*. *The strategies firms adopt in this SPE are:*

$$\sigma_1(t) = \begin{cases} 1 & \textit{if} \quad t = t^* \ \ \textit{or}\ \ [t > t^* \ \ \& \ \ A(t) = \{t\}], \\ 0 & \textit{otherwise}; \end{cases}$$

$$\sigma_2(t) = \begin{cases} 1 & \textit{if} \quad t = t^* \ \ \textit{or}\ \ [t > t^* \ \ \& \ \ L(t) \geq F(t) \ \ \& \ \ A(t) = \{t\}], \\ 0 & \textit{otherwise}. \end{cases}$$

Proof: See Appendix A.

Here, if both firms are entering at t^*, there is no gain from a unilateral deviation to enter earlier, as the payoff to a leader at t^* is greater than a leader's payoff from entry at any earlier date; this follows from the way set B is constructed (*NPL*). Similarly, there is no gain from deviating and entering later; the set B is constructed so that the payoffs to the leader and the follower are equal (*RE*). Specifically, if one firm deviates it will get the same payoff – the payoff of the follower $F(t^*)$ rather than the payoff associated with attempted entry as a leader, $(L(t^*) + F(t^*))/2$.

The equilibrium strategies of both firms are to wait before t^*, enter at t^*, and for both firms to adopt the strategies specified for the LSPE off-the-equilibrium path (that is, for $t > t^*$). It is worth noting that if $L(t^*) = F(t^*)$ in the LSPE of the game, set B also includes the leader's preferred subgame perfect equilibrium.

Below, we will outline how to determine the total number of pure-strategy SPE with unique leader entry times in the game. For simplicity we restrict our attention to the situations where there is a finite number of SPE with RE.

Now let us turn our attention to another potential SPE at $t = 0$.

Lemma 3. *If* $L(0) > F(0)$, *there is an SPE in which both firms enter at* $t^* = 0$. *The strategies firms adopt in this SPE are:*

$$\sigma_1(t) = \begin{cases} 1 & \textit{if} \quad t = 0 \ \ \textit{or}\ \ [t > 0 \ \ \& \ \ A(t) = \{t\}], \\ 0 & \textit{otherwise}; \end{cases}$$

$$\sigma_2(t) = \begin{cases} 1 & \textit{if} \quad t = 0 \ \ \textit{or}\ \ [t > 0 \ \ \& \ \ L(t) \geq F(t) \ \ \& \ \ A(t) = \{t\}], \\ 0 & \textit{otherwise}. \end{cases}$$

Proof: See Appendix A.

In a similar manner to the case in Lemma 2, there is no gain from unilaterally deviating and entering later as $L(0) > F(0)$. The equilibrium strategies of both firms are to enter at $t = 0$ and adopt the strategies specified for the LSPE off-the-equilibrium path (that is, for $t > 0$). Note that this potential equilibrium is explicitly ruled out by Fudenberg and Tirole (1985), as they assume that the follower's payoff is greater than the leader's at the start of the game. Furthermore, this equilibrium can be the LSPE if $A(0)$ is an empty set.[13]

Again, we are also interested in the total number of pure-strategy SPE. Let n_0 be the number of SPE of the game that have immediate entry at $t = 0$. Utilizing the *Iverson bracket*, which takes a value of 1 if the condition specified is satisfied and 0 otherwise, $n_0 = [L(0) > F(0)]$.

Now, we discuss equilibria with a second-mover advantage. To do this, let $\{U_s\}_{s=1}^k$ be the connected components of $\{t \in [0, t^{max}], L(t) \leq F(t)\}$, where k is the smallest integer. Clearly k is finite because L and F are continuous on a compact $[0, t^{max}]$. With this representation, we are in a position to present a lemma characterizing all SPE of the game with a second-mover advantage.

Lemma 4. *For any region U_s, $s = 1, \ldots, k$, apply Lemma 1 to find the LSPE of the region with the leader's entry time at t^*. If $t^* \in A(0)$ and the equilibrium is not a RE ($t^* \notin B$) equilibrium, it is a second-mover advantage equilibrium for the entire game. The strategies firms adopt in this SPE are:*

$$\sigma_1(t) = \begin{cases} 1 & \textit{if} \quad t = t^* \ \textit{or} \ [t > t^* \ \& \ L(t) \geq F(t)] \ \textit{or} \ [t > t^* \ \& \ A(t) = \{t\}], \\ 0 & \textit{otherwise}; \end{cases}$$

$$\sigma_2(t) = \begin{cases} 1 & \textit{if} \quad t > t^* \ \& \ L(t) \geq F(t), \\ 0 & \textit{otherwise}. \end{cases}$$

Proof: See Appendix A.

In the equilibria described in this lemma, the leader invests at t^* as there is no gain from investing earlier because $t^* \in A(0)$. There is also no gain from investing later as: the equilibrium is the LSPE for a given region U_s; and both firms enter whenever $L(t) \geq F(t)$ for $t > t^*$, ensuring entry cannot be postponed until after this region. Here, in a similar manner to the situation described in Lemma 1, the firms have different strategies to allow for asymmetries in the equilibria. In these equilibria the leader receives a lower payoff than the follower. As a result, the follower also has no incentive to deviate. Note that similar second-mover advantage equilibria are present in Dutta et al. (1995) and Hoppe and Lehmann-Grube (2005).

As noted above, focusing on the leader's entry time as the defining feature of a unique equilibrium, we can determine the number of pure-strategy SPE in the game. To aid in counting of the number of pure-strategy SPE, let: (i) $n_{B \setminus L}$ be the number of RE SPE that are not also LSPE of any region U_s; and (ii) n_{NA} be the number of

[13] As a point of clarification, there is an equilibrium when $L(0) = F(0)$. This equilibrium is not captured here; rather, it is included in set B, as described in Lemma 2.

LSPE corresponding to each respective region U_s that do not belong to $A(0)$. From Lemmas 3 and 4, the total number of these two types of SPE (RE and second-mover advantage equilibria) is $k + n_{B\backslash L} - n_{NA}$.

So far we have detailed the conditions for an LSPE, preemption equilibria with immediate entry, and equilibria with rent equalization and a second-mover advantage. Using the Lemmas presented above, we are now in the position to summarize all the SPE of the game, and this is done in the following proposition.

Proposition 1. *Lemmas 2, 3 and 4 characterize all of the SPE in the continuous-payoff timing game. The number of pure-strategy SPE with unique leader entry times* $n = n_0 + k + n_{B\backslash L} - n_{NA}$.

Proof: See Appendix A.

As described in Proposition 1, our technique allows for the characterization of all SPE in the continuous entry game with two firms. Lemma 2 outlines the equilibria in which there is rent equalization. There is a first-mover advantage in the SPE described in Lemma 3; there will be immediate joint adoption at $t = 0$. Finally, Lemma 4 describes equilibria in which there is a second-mover advantage. Note that the equilibria detailed in Lemmas 2, 3 and 4 are mutually exclusive. However, the LSPE described by Lemma 1 is covered by one of Lemmas 2, 3 or 4.

At this point turn our attention to the formula that counts the number of pure-strategy SPE with unique leader entry times. Whenever the leader's payoff at the start of the game exceeds that of the follower, there is an equilibrium with immediate entry. In addition, there are equilibria associated with all regions, except for the regions that are not part of $A(0)$. Finally, there could be RE SPE that are not LSPE of any region. We illustrate how the formula can be applied in Section 3.1.

We now consider some of the implications of the Proposition. First, let us focus on entry games with multiple SPE, when one of the equilibria is the SSPE. In the SSPE both firms receive a higher payoff than in any other SPE. For the SSPE to exist it must be feasible to Pareto rank all equilibria. This will be possible, for sure, when there are no second-mover advantage equilibria (as described in Lemma 4) or when there is a unique second-mover advantage equilibrium and it is also the LSPE of the game (detailed in Lemma 1). If only rent equalization or immediate-entry equilibria exist, they are directly comparable. This is not necessarily true with a second-mover advantage equilibrium; one firm could be better off while the other is worse off compared with alternative SPE. Only when the second-mover advantage equilibrium is unique and provides the leader with its highest possible payoff can we be sure that a Pareto ranking is feasible. Note, it is possible to determine that the SSPE, provided it exists, is the LSPE of the game. This means that the SSPE is the SPE with the latest possible entry time. No other SPE can be the SSPE of the game as the LSPE provides the leader with their highest payoff. This is summarized in the following corollary.

Corollary 1. *If the equilibria can be ranked, the SSPE is the LSPE. A sufficient condition for the SSPE to exist is that: (i) there are no second-mover advantage* *equilibria; or (ii) there is a unique second-mover advantage equilibrium that is also the LSPE.*

Proof: The proof follows from the discussion above.

Moving away from games with multiple equilbria, let us now consider the economically important situations in which there is a unique pure-strategy SPE. Uniqueness aids welfare comparisons and simplifies empirical investigations of market entry. Our model has the advantage of outlining sufficient conditions required to ensure uniqueness.

Corollary 2. *If the equilibrium is unique, it is both the LSPE and the SSPE. The first firm's stopping time t^* is given by equation 2 and the leader and follower strategies by 3 and 4. A sufficient condition for a unique SPE is that:*

1. *$L(t)$ is non-increasing;*
2. *$F(t)$ is non-increasing; or*
3. *$L(t) < F(t)\ \forall\ t < t'$ and $L(t) > F(t)\ \forall\ t > t'$, where $t' \in [0, T]$.*

Proof: The proof follows from Proposition 1.

Working through the three parts of the corollary, the intuition is as follows. Part (1) of Corollary 2 states that there will be a unique SPE with immediate entry if $L(t)$ is non-increasing. With a non-increasing payoff, there is no advantage to the leader from delaying entry; there is a unique SPE with immediate entry. From a practical stand point, this result has the advantage that uniqueness does not rely on the shape of the follower's payoff function when the leader's payoff function is non-increasing.

Turn our attention now to Part (2). Hoppe and Lehmann-Grube (2005) analyzed this case focusing on the situation when $F(0) > L(0)$. This is an economically important scenario – as highlighted by Hoppe and Lehmann-Grube (2005) – because many entry games involve a payoff to the follower that is either decreasing (or non-increasing) with respect to the time. For example, in situation in which technology is improving with time, earlier leader entry advantages the follower as the leader goes to market with a less advanced product or production process.

Considering this case, the unique equilibrium can be further simplified to be represented by the leader's entry t^* as follows:

$$t^* = \min \arg\max_t \min[L(t), F(t)]. \tag{6}$$

Given that the followers payoff is non-increasing, delaying entry after the payoffs to the leader and the follower intersect for the first time is never optimal. As a result, the unique SPE will involve either: rent equalization if there is a time where $L(t)$ and $F(t)$ intersect and $L(t)$ is at its historic maximum for the game up until that time; or, alternatively, a second-mover advantage.

There is also the possibility that $F(t)$ is non-increasing and $F(0) \leq L(0)$. At $t = 0$, as the leader's payoff is higher than the follower's, there will be a unique preemption equilibrium with immediate entry. Moreover, this holds regardless as to the shape of $L(t)$. As the payoff to the follower is never any higher than it is at $t = 0$, if the leader and follower curves ever interest it cannot be at a higher level than what the leader could receive from immediate entry – there will be an incentive to deviate and enter immediately.

Now consider Part (3) of the Corollary. There are three situations that are consistent with this scenario. First, if $t' = 0$, it is the case that $L(t) > F(t)$ $t \in (0, T]$ and $L(0) \geq F(0)$; there will be immediate entry in a preemptive equilibrium with each firm vying to be the leader in the market. An example of this would be entry into a natural monopoly; once more, the incentive to preempt ensures that there is immediate entry, potentially dissipating all or some of the available monopoly rents.

Second, if $t' = T$, the follower's payoff dominates the leader's for every time up until the end of the game; that is, $F(t) > L(t)$ $t \in [0, T)$ and $F(T) \geq L(T)$. In this case, while both firms would prefer to be the follower in this second-mover advantage game, the firm with the relatively weaker bargaining position will enter at the best time for the leader.

Third, if $t' \in (0, T)$ there is a unique time at which the initial follower-advantage is reversed and the payoff to the leader exceeds that to the second entrant. The unique SPE will involve either: rent equalization if $t^* = t'$; or, alternatively, a second-mover advantage if $t^* < t'$. Note that here we do not place any restrictions on the shape of the $L(t)$ and $F(t)$ functions, other than the requirement that there is a unique change from a follower to a leader advantage.

There are many economic situations in which there is a one-time change from a follower to a leader advantage. For example, if entry costs reduce over time, the leader's payoff relative to that of the follower could monotonically improve over time, reversing an initial follower advantage. A similar relationship between follower and leader payoffs could hold in a product-innovation model in which the quality improves with later entry (see the product-innovation example presented below in Section 3.1). It also might apply to situations when the benefits of free riding diminish after a period of delayed entry.

3.1 Examples

To provide some further intuition for the main results in the paper, and to allow for a closer comparison to the previous literature, we construct three examples. The first two are modifications of the process- and product-innovation timing examples of Hoppe and Lehmann-Grube (2005). Essentially, we augment their examples to allow for an experience effect or switching cost for consumers. This alters consumers' incentive to switch supplier when there is entry; this setup can generate non-monotonic payoff functions for both the leader and the follower. Our third asset-market example is adapted from Dutta and Rustichini (1993). We completely characterize all SPE of these games.

Consider the following setup for the first two examples. Two firms are contemplating entering a market at some time $t_i \in [0, \infty)$ for $i = 1, 2$. The first firm that enters gets an instantaneous flow of monopoly profit R_m until the time when the second firm enters, which is optimally chosen by the second firm. After entry by the second firm, they share the market in proportions (R_1, R_2). We assume that the market exists for an infinite period of time. We also assume, for simplicity, that each firm's R&D costs per unit of time are zero ($k(t) = 0$ in the terminology of Hoppe and Lehmann-Grube (2005)). The payoffs are discounted by a common discount factor $e^{-r\tau}$, so that the net-present value of profits for the leader entering at t_1 and follower entering at t_2 are:

$$\pi_1(t_1, t_2) = \int_{t_1}^{t_2} e^{-r\tau} R_M(t_1) d\tau + \int_{t_2}^{\infty} e^{-r\tau} R_1(t_1, t_2) d\tau; \tag{7}$$

and

$$\pi_2(t_1, t_2) = \int_{t_2}^{\infty} e^{-r\tau} R_2(t_1, t_2) d\tau. \tag{8}$$

Example 1: Process innovation with an experience effect. In the process-innovation game, the production technology a firm can use when it enters the market improves over time, allowing for a lower marginal cost with later entry. We assume that a firm adopts the best technology available when it enters, and that it uses this technology from thereon. This means that a firm entering later has a lower cost. Specifically, marginal costs decrease over time according to the cost function $c_i(t) = e^{-\alpha t_i}$, where $\alpha > 0$ is the rate of technological progress. The market demand in each period is 1 unit at a constant price of 1. Given these assumptions, the per-period monopoly profit is $R_M = 1 - c_1$.

We augment this basic process-innovation game by assuming that after both firms enter they share the market in proportions $(s(t_2 - t_1), 1 - s(t_2 - t_1))$, where $s(t) = 1 - 0.5e^{-\beta t}$. This functional form allows for an *experience effect*; the longer the leader operates alone, the larger its share of the market after entry by the follower, where $\beta > 0$ measures the strength of this effect.[14] Given this, with both firms in the market, the duopoly profits are

$$R_1 = (1 - c_1)s(t_2 - t_1), \;\; R_2 = (1 - c_2)(1 - s(t_2 - t_1)).$$

Herein lies the tradeoff for the firms when deciding their optimal entry times. Early entry – if they manage to do so before their rival – helps a firm to develop a captive customer base. Later entry, on the other hand, allows a firm to enter the market with lower production costs.

To simplify the analysis we assume that $r = 1$ and that $\alpha = 1$. This lets us explicitly derive the payoffs[15]:

[14] Note, Simon and Stinchcombe (1989, pp. 1175-1178) consider a market-entry game with loyal customers that has similarities to this experience-effect example.

[15] For ease of exposition we state the payoffs in terms of c_1 and $\hat{c}_2$, rather than t_1 and $\hat{t}_2$.

$$L(t_1) = \begin{cases} (1-c_1)c_1(1-0.5(\hat{c}_2/c_1)^{\beta+1}) & \text{if} \quad t_1 < \hat{t_2}, \\ 0.5(1-c_1)c_1 & \text{if} \quad t_1 \geq \hat{t_2}. \end{cases} \tag{9}$$

and

$$F(t_1) = \begin{cases} 0.5(1-\hat{c}_2)\hat{c}_2(\hat{c}_2/c_1)^{\beta} & \text{if} \quad t_1 < \hat{t_2}, \\ 0.5(1-c_1)c_1 & \text{if} \quad t_1 \geq \hat{t_2}, \end{cases} \tag{10}$$

where $c_i(t) = e^{-t_i}$, $\hat{c}_2 = \frac{1+\beta}{2+\beta}$ and $\hat{t_2} = \ln\left(\frac{2+\beta}{1+\beta}\right)$.

Several points are worth noting here. First, the follower's payoff as a function of entries of both firms is $F(t_1, t_2) = 0.5(1-c_2)c_2(c_2/c_1)^{\beta}$. Given that this function is separable in c_1 and c_2, the first order conditions with respect to c_2 give a unique value of $\hat{c}_2$. This means that for the follower the optimal time of entry is $\hat{t_2}$, or t whenever t exceeds $\hat{t_2}$.

Second, in Figure 1 we construct three curves when $\beta = 0.5$: $L(t_1)$ and $F(t_1)$ given by equations 9 and 10 respectively, but also $M(t_1) = 0.5(1-c_1)c_1$ that represents payoff of either firm when they both enter simultaneously at t_1. Given the previous point regarding the follower's optimal entry strategy, all curves need to coincide after time $\hat{t_2}$. $M(t_1)$ is lower than $F(t_1)$ prior to $\hat{t_2}$, because before this time joint adoption is not the follower's optimal entry strategy. Later entry by the follower can have a positive spillover onto the leader, helping increase $L(t_1)$ above $F(t_1)$ (in the Figure this occurs between t^* and $\hat{t_2}$).

Third, let us now apply our technique to solve the model. Figure 1 shows that in this case both payoff functions $L(t_1)$ and $F(t_1)$ are non-monotonic, with both curves increasing and decreasing over time. Applying the method outlined in Proposition 1 we can show that there is a unique SPE. It is evident that there is no equilibrium with immediate entry ($n_0 = 0$). There is only one region $[0, t^*]$, so that $k = 1$; values above $\hat{t_2}$ are all below the historical maximum of the leader's payoff. The rent equalization SPE with entry time t^* is also the LSPE of the first and only region, so $n_{B\setminus L} = 0$. Finally, the LSPE of this region belongs to $A(0)$, so $n_{NA} = 0$. As specified in the Proposition, we sum these values to derive $n = 1$.

One can see that $A(0) = [0, t^*]$ and $B = \{t^*\}$. This is because for all times between 0 and t^* the payoff to the leader is increasing, while it is still less than the payoff to the follower. At t^*, the leader and follower's payoff coincide – at this point there is a preemption equilibrium. The equilibrium strategies are for both firms to enter at any $t \geq t^*$.

Fourth, we can also describe how the equilibria change in response to a change in the importance of the experience effect, as measured by β. We need to compare the maximum possible payoff of $L(t_1)$ before $\hat{t_2}$ with the highest joint adoption $\max M(t_1) = M(t_M) = 1/8$. The leader's payoff as a function of the entry times of both firms is the function $L(t_1, t_2) = (1-c_1)c_1(1-0.5(c_2/c_1)^{\beta+1})$. This function is a decreasing function in both β and c_2, while the follower's optimal choice $\hat{c}_2$ is increasing in β. As a result, the maximum possible payoff of $L(t_1)$ is monotonically decreasing in β. In this example, there is a critical value of $\beta^* \approx 0.7$; if $\beta < \beta^*$, as the peak of the $L(t_1)$ exceeds the maximum payoff possible with joint adoption

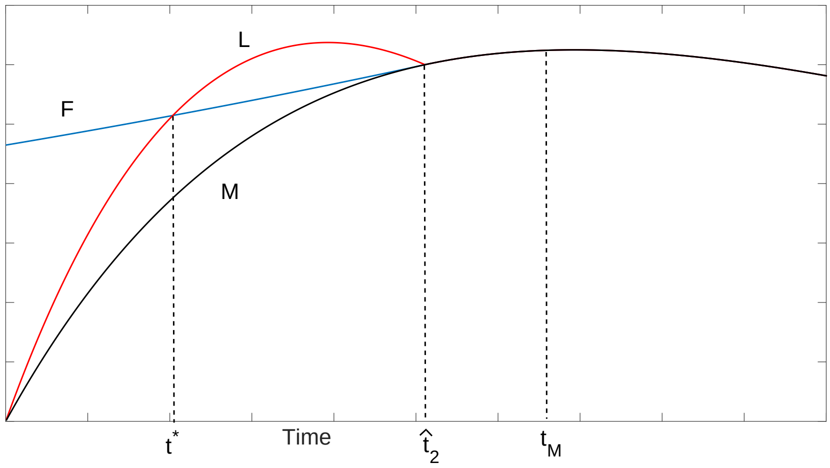

Figure 1: Non-monotonic $L(t)$ and $F(t)$ payoff functions in a process-innovation game

$M(t_M)$, there is a unique preemption equilibrium. This is the case highlighted in Case A in Fudenberg and Tirole (1985). If, however, $\beta > \beta^*$ the peak of $L(t_1)$ before time $\hat{t}_2$ is less than $M(t_M)$, there will be additional rent equalization equilibria with possible entry before and at t_M. This is Case B in Fudenberg and Tirole (1985). It is noteworthy that here we derive these two cases from the primitives of the model, whereas Fudenberg and Tirole (1985) effectively assume that the firms will always enter simultaneously after $\hat{t}_2$.

The intuition underlying the role of β is also worth further comment. As mentioned previously, later entry by the follower can have a positive spillover onto the leader, helping increase $L(t_1)$. With a stronger experience effect (a higher β) the follower optimally enters earlier, which in turn decreases this spillover on the leader, decreasing the maximum possible payoff of $L(t_1)$. Hence, an increase in β decreases the maximum of $L(t_1)$ prior to $\hat{t}_2$ relative to $M(t_M)$, which could lead to the case of multiple equilibria.

Example 2: Product innovation with switching costs. In this model, the potential quality of the product a firm can take to market improves over time; for example, the quality of a phone handset will typically improve the longer a firm waits to launch it. In a similar way as to the process-innovation model above, when a firm enters the market, they sell a product of the highest quality available at the time. Note that, this is a one-off decision – firms sell the same quality product from their time of entry, until the end of the game. As a result, waiting is advantageous as it allows a firm to sell a better quality product.

Following Dutta et al. (1995) and Hoppe and Lehmann-Grube (2005), consumers value quality in a vertically-differentiated goods model. We assume that the quality of a product, denoted by s, is increasing monotonically over time according to the function $s = t$. For simplicity, it is assumed that R&D and production costs are zero and independent of quality.

Like in Tirole (1988), preferences differ according to a taste parameter θ, where θ is uniformly distributed between $[0,1]$. Each consumer has a unit demand for the good and has utility of $U = s_i\theta - p_i$, where s_i and p_i are the quality and price offered by firm i. A consumer will buy at most one unit from a firm provided that $U \geq 0$ for that product and, if there are more than one firm in the market, the consumer will buy one unit from the firm that provides her with the highest net utility (again provided that $U \geq 0$).[16] In this framework $F(t)$ is monotonically decreasing function. As product quality is improving with the time of entry, the follower's competitiveness is reduced the later the leader comes into the market.

To this standard framework, we introduce a switching cost for consumers that have had experience with a particular good. Specifically, a consumer that has been serviced by firm 1 for the period of time τ will require an additional utility of at least $E(\tau) = \beta\tau^2$ if she is to have an incentive to switch to firm 2, where β is relative importance of switching costs. Consequently, taking each entry time as given at t_1 and t_2, respectively, there will be a consumer with a taste parameter $\theta = \theta_2$ who is just indifferent between switching from buying the leader's product to changing over to buy the second entrant's offering. That is, θ_2 solves

$$\theta_2 t_1 - p_1 + E(t_2 - t_1) = \theta_2 t_2 - p_2.$$

There will also be a consumer with a taste parameter θ_1 who is just indifferent between buying from the leader and not buying at all. In other words, for this indifferent consumer, θ_1 solves

$$\theta_1 t_1 - p_1 = 0.$$

Furthermore, if switching between providers is to occur, it will happen only at the point in time at which the second firm enters, and not at a later date.

For this model, the instantaneous monopoly profit is $R_M = t_1/4$, while the instantaneous duopoly profits are

$$R_1 = \frac{(t_2 - t_1 + E(t_2 - t_1))^2 t_1 t_2}{(4t_2 - t_1)^2 (t_2 - t_1)}, \quad R_2 = \frac{(2(t_2 - t_1)t_2 + E(t_2 - t_1)(t_1 - 2t_2))^2}{(4t_2 - t_1)^2 (t_2 - t_1)},$$

given the entry times are t_1 and t_2, respectively.

To simplify the analysis we assume that $r = 1$ and $\beta = 0.5$. Figure 2 shows that in this case the payoff functions of both the $L(t)$ and $F(t)$ are non-monotonic. Of particular importance is the fact that the switching cost introduced here generates a non-monotonic payoff for the follower. This has the following intuition. The longer a consumer buys from one firm, the greater her cost of switching to buy the product from the other firm; this provides an incentive to the follower to enter the market earlier to reduce the size of the leader's captive market. Consequently, early entry by the leader elicits earlier – inefficient – entry by the follower limiting the quality

[16]See Hoppe and Lehmann-Grube (2001) and Hoppe and Lehmann-Grube (2005) for more details. Note that, while the choice of price is not dynamically optimal, we adopt this framework to aid comparison to the example in Hoppe and Lehmann-Grube (2005).

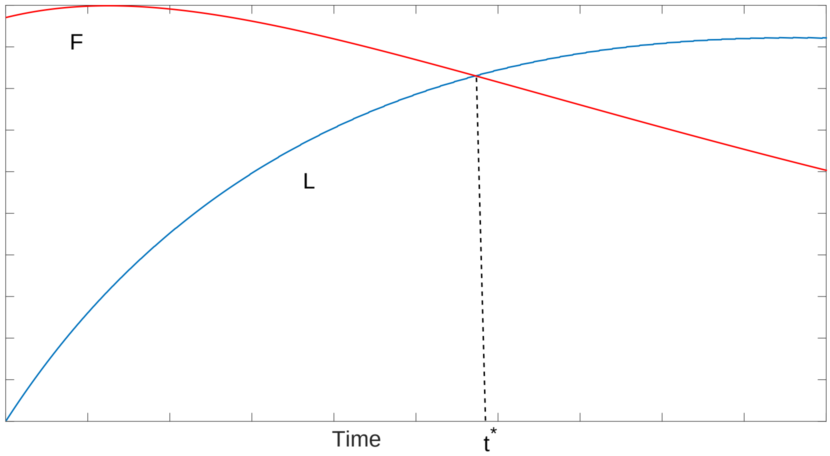

Figure 2: Non-monotonic payoff functions for $F(t)$ and $L(t)$ in a product-innovation game

of the product it takes to market. It can be the case, as shown in Figure 2, that this inefficiency is sufficiently strong to lead to a non-monotonicity in the follower's payoff.

Let us use the algorithm developed in this paper to derive the SPE in this product-innovation example. Note, that the conditions for a unique SPE outlined in Corollary 2, Part (3) hold in this example: $F(t) > L(t)\ \forall\ t < t^*$ and $L(t) > F(t)\ \forall\ t > t^*$. In other words, there is a one-time reversal from an initial follower advantage to a leader advantage at time t^*. In this example $A(0) = [0, t^*]$ and $B = \{t^*\}$. As shown in Figure 2 the $L(t)$ and $F(t)$ curves intersect once at t^*. Consequently, the unique SPE involves preemption with joint entry at t^*. The equilibrium strategies are for both firms to enter at any $t \geq t^*$.

Example 3: Asset sales. When should a trader sell an asset? A vendor making this decision will have to take into account the actions of other sellers. Following Dutta and Rustichini (1993), we consider two potential sellers of an asset in a market with the following features. First, the price of the asset is appreciating, perhaps representing the case when the market demand for the asset increases over time. Second, the follower's sale price is negatively affected if the other party sells their asset first. A possible example of the payoffs to the first seller, shown by $L(t)$, and the second seller, $F(t)$, is illustrated in Figure 3. As before, both payoffs are functions of the leader's time of sale t.

Let us utilize Proposition 1 to count the number of unique SPE. First, as the leader's payoff does not exceed the follower's at time $t = 0$, $n_0 = 0$. Second, there are two regions ($k = 2$) with a follower advantage, $[0, t_1]$ and $[t_2, t_3]$. Third, there is one rent-equalization equilibrium with leader entry at t_2 that is not also an LSPE of either region, so that $n_{B\backslash L} = 1$. Note, this type of equilibrium did not exist in our previous two examples. Fourth, the LSPE of both regions are also equilibria of the entire game, hence $n_{NA} = 0$. In sum this means that there are three unique SPE.

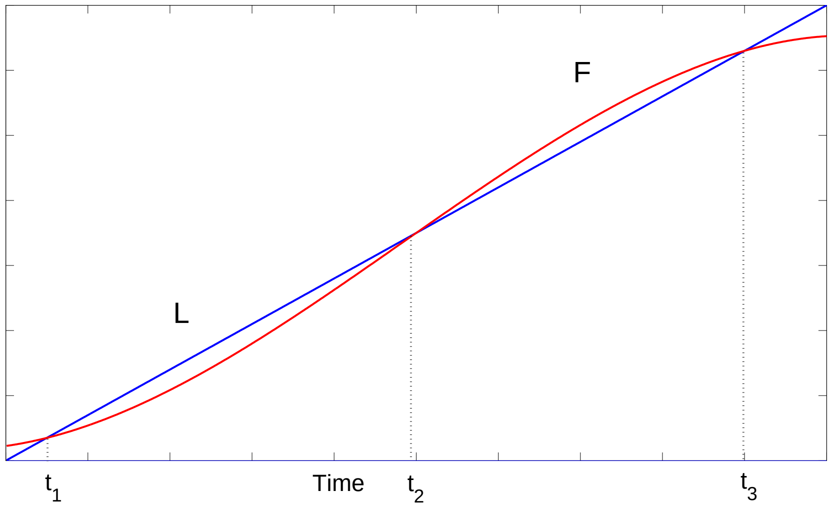

Figure 3: Asset sales with increasing potential sale prices

In this asset-market example, $A(0) = [0, t_1] \cup [t_2, t_3]$. Set B contains $\{t_1\}$, $\{t_2\}$, and $\{t_3\}$; consequently, the rent-equalization equilibria involve entry at t_1, t_2 or t_3, with the last of these equilibria being the Pareto preferred SSPE. These are the only pure-strategy equilibria of the game. The equilibrium strategies that support preemptive entry at t_3^* are for both firms to enter at any $t \geq t_3^*$. The equilibrium strategies that support t_2^* are for both firms to enter at t_2^* and at any $t \geq t_3^*$. The preemptive equilibrium with entry at t_1 requires both firms to enter when $t \in [t_1, t_2]$ and for any $t \geq t_3$.

4 Discontinuous payoffs

As noted in the introduction, discontinuities in payoffs arise in many economic situations. We turn our attention to this issue now. To do this we make an assumption regarding the nature of these discontinuities. We assume that all functions are right-continuous; that is, all functions have no break when the limit point is approached from the right. Given the sort of structural breaks that are likely to arise in timing games, this seems like the most natural assumption to make; for example, an action by a third party in a related market could result in a discontinuous jump (up or down) in the payoff from innovating in the market of interest. Similarly, when selling an asset, a sale by one party could have a discontinuous effect on the potential sale price for the second vendor. Moreover, right-continuous functions are consistent with the (always present) discontinuity at $t = 0$.[17]

[17]Note that the n-player game ($n > 2$) differs from the 2-player game in that discontinuities can arise with three or more players because a follower's response need not be continuous in the leader's entry time (Fudenberg and Tirole, 1985). That means that provided we can identify the unique SPE

We assume that there is a finite number of discontinuities and introduce the following set D that contains all times at which either the leader's or follower's payoff function is discontinuous. Specifically,

$$D = \{ \ t \ | \ \lim_{\tau \to t^-} L(\tau) \neq L(t) \ \text{ or } \ \lim_{\tau \to t^-} F(\tau) \neq F(t) \ \text{ or } \ t = 0 \ \}. \tag{11}$$

It is worth noting that $t = 0$ is also included in this set D as it has similar properties to other elements of this set, in that limits with $\tau \to 0^-$ are not defined.

To find the set of SPE we adapt the technique developed in Section 3. A crucial proviso here is that we need to ensure that an equilibrium exists; for example, non-existence could be an issue if the set $A(0)$ does not contain its supremum. To explore this, first consider the case when $F(t)$ is discontinuous but the supremum is not in set $A(0)$. This situation is illustrated in Figure 4a. Note that in this case $A(0) = [0, t')$. Consequently, the leader wants to enter before t' but as close to this time as possible; no pure-strategy equilibrium exists.

Second, with a discontinuous $L(t)$ it is also possible that the supremum does not belong to set $A(0)$ itself. We illustrate this situation in Figure 4b. One can see that $A(0) = [0, t')$. The equilibrium does not exist in this example because the leader would like to enter before t', but as close as possible to this time.

To proceed, utilizing Assumption 1, let there be a small length of time ε just prior to the discontinuity in payoffs that represents the last time before the discontinuity that a firm can enter, as outlined below.

Assumption 6. *The minimum time before a discontinuity that a firm can opt to enter the market is ε, where $\varepsilon > 0$.*

This minimum time ε before the discontinuity is effectively the last 'period' in which a firm can enter prior to the break in payoffs. This assumption effectively ensures that the presence of a discontinuity does not result in the non-existence of equilibria.

Now we can modify the method developed in Section 3 to accommodate for payoff functions with discontinuities. Again, if a point of discontinuity is to be a time at which a firm enters as part of an SPE, it must be the case that neither the leader nor the follower wishes to preempt by entering before the discontinuity. To capture this, we augment the *NPL* condition for points of discontinuity:

Condition 4. *No Preemption by the Leader at a Discontinuity (NPLD):*
$(L(t^*) + F(t^*))/2 > L(\tau) \ \forall \ \tau < t^*$.

To ensure we capture all SPE at points of discontinuity, we introduce an additional condition:

Condition 5. *Leader advantage (LA):* $L(t^*) > F(t^*)$.

for all $(n-1)$ player subgames, the n-player game is qualitatively equivalent to the 2-player game with discontinuities.

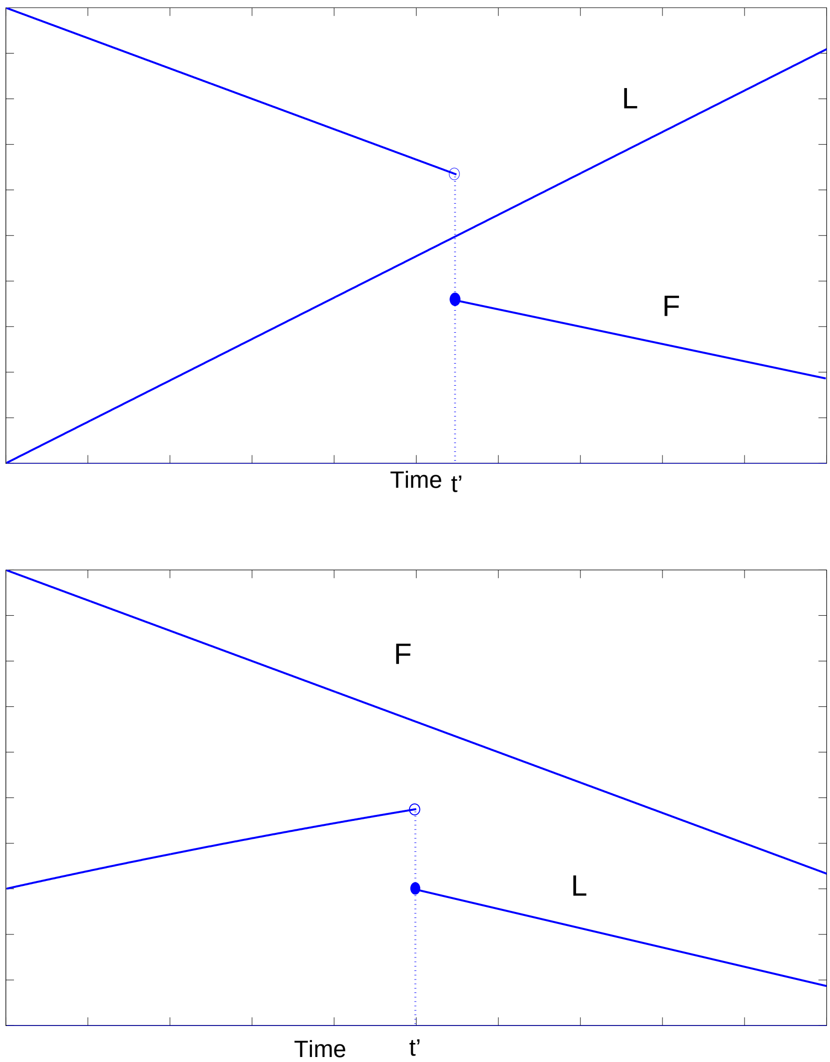

Figure 4: Discontinuities in the payoff functions

To solve for SPE at points of discontinuity, we introduce set C that contains all points satisfying both condition *NPLD* and *LA*, that is:

$$C(t') = \{t \geq t', t \in D |\ L(t) > F(t)\ \&\ (L(t) + F(t))/2 > L(\tau)\ \forall\ \tau \in [t', t)\}. \quad (12)$$

Here, in an analogous way to set A, the leader would not wish to preempt by entering prior to the discontinuity, as their expected payoff from entering at the discontinuity (and possibly becoming the follower) is still higher than entering at any previous point in time (*NPLD*). Furthermore, note that some discontinuities could be included in $A(0)$, provided the conditions in (1) are satisfied. By considering the *NPLD* and *LA* conditions for points of discontinuity we are able to consider potential entry times where there are discontinuities and $L(t) > F(t)$; these situations are not described by $A(0)$. Furthermore, as pointed out earlier when $A(t')$ was defined, the *NPLD* condition does not apply if $t = t'$. In particular, this means that for $t = 0$ to be contained in $C(0)$ only requires that $L(0) > F(0)$.

We are now in a position to present a lemma, that modifies Lemma 1 to accommodate discontinuous payoff functions.

Lemma 5. *The first firm's stopping time in the LSPE, which always exists, is:*

$$t^* = \arg\max_t [A(0) \cup C(0)]. \quad (13)$$

The strategies firms adopt in the LSPE are:

$$\sigma_1(t) = \begin{cases} 1 & \textit{if} \quad [A(t) \cup C(t)] = \{t\}, \\ 0 & \textit{otherwise}; \end{cases}$$

$$\sigma_2(t) = \begin{cases} 1 & \textit{if} \quad L(t) \geq F(t)\ \&\ [A(t) \cup C(t)] = \{t\}, \\ 0 & \textit{otherwise.} \end{cases}$$

Now, the LSPE could occur at a point of discontinuity; all points with $L(t) \leq F(t)$ – including discontinuities – are covered by $A(0)$, whereas discontinuities with $L(t) > F(t)$ are covered by $C(0)$. Consequently, the set $A(0) \cup C(0)$ covers all possible SPE arising both at continuous and discontinuous points. In contrast to Lemma 1, the time $t = 0$ will necessarily belong to either set $A(0)$ or $C(0)$, meaning that non-existence when $A(0) = \emptyset$ is no longer an issue.

The following lemma modifies Lemma 2 to accommodate discontinuous payoff functions in the rent-equalization equilibria. As illustrated, there are very few changes from Lemma 2, except for the firms' off-equilibrium strategies.

Lemma 6. *For any $t^* \in B$ there is a corresponding SPE in which both firms enter at t^*. The strategies firms adopt in this SPE are:*

$$\sigma_1(t) = \begin{cases} 1 & \textit{if} \quad t = t^* \ \textit{or}\ (t > t^*\ \&\ [A(t) \cup C(t)] = \{t\}), \\ 0 & \textit{otherwise}; \end{cases}$$

$$\sigma_2(t) = \begin{cases} 1 & \text{if} \quad t = t^* \quad or \quad (t > t^* \ \& \ L(t) \geq F(t) \ \& \ [A(t) \cup C(t)] = \{t\}) \,, \\ 0 & \text{otherwise.} \end{cases}$$

Proof: See Appendix A.

Lemma 3 details the equilibrium when there is immediate entry. Below, we generalize this result to all points of discontinuity that belong to $C(0)$.

Lemma 7. *For any $t^* \in C(0)$ there is a corresponding SPE in which both firms enter at t^*. The strategies firms adopt in this SPE are:*

$$\sigma_1(t) = \begin{cases} 1 & \text{if} \quad t = t^* \quad or \quad (t > t^* \ \& \ [A(t) \cup C(t)] = \{t\}) \,, \\ 0 & \text{otherwise;} \end{cases}$$

$$\sigma_2(t) = \begin{cases} 1 & \text{if} \quad t = t^* \quad or \quad (t > t^* \ \& \ L(t) \geq F(t) \ \& \ [A(t) \cup C(t)] = \{t\}) \,, \\ 0 & \text{otherwise.} \end{cases}$$

Proof: See Appendix A.

Next, we consider equilibria with a second-mover advantage. To do this, let $\{U_s\}_{s=1}^k$ be the connected components of $\{t \in [0, t^{max}], L(t) \leq F(t)\}$, where k is the smallest integer. As there is a finite number of discontinuities, k is also finite.[18] Note that if a discontinuity that belongs to region U_s does not involve a reversal of the relative payoffs of leader and follower, it will not be a boundary of that region. With this representation, we are in a position to present a lemma characterizing all SPE with a second-mover advantage in the game with discontinuous payoffs.

Lemma 8. *For any region U_s, $s = 1, \ldots, k$, apply Lemma 5 to find the LSPE of the region, denoted as t^*. If $t^* \in A(0)$ and this equilibrium is not a RE ($t^* \notin B$), it is a second-mover advantage equilibrium for the entire game. The strategies firms adopt in this SPE are:*

$$\sigma_1(t) = \begin{cases} 1 & \text{if} \quad t = t^* \quad or \quad (t > t^* \ \& \ L(t) \geq F(t)) \quad or \quad (t > t^* \ \& \ [A(t) \cup C(t)] = \{t\}) \,, \\ 0 & \text{otherwise;} \end{cases}$$

$$\sigma_2(t) = \begin{cases} 1 & \text{if} \quad t > t^* \ \& \ L(t) \geq F(t), \\ 0 & \text{otherwise.} \end{cases}$$

Proof: See Appendix A.

In a similar manner to Lemma 4, here we do not need to consider potential equilibria at points of discontinuity that are covered by set $C(0)$ as they are not included in any region U_s by definition.

Now we generalize Proposition 1 to accommodate discontinuities.

[18] Set D divides $[0, t^{max}]$ into a finite number of areas separated by discontinuities in $L(t)$ and $F(t)$. Within each area, because $L(t)$ and $F(t)$ are continuous there are a finite number of regions where $L(t) \leq F(t)$. Consequently, over $[0, t^{max}]$ there are a finite number of regions where $L(t) \leq F(t)$; hence k is bounded above by a finite number.

Proposition 2. *Lemmas 6, 7 and 8 characterize all of the SPE when payoffs can be discontinuous.*

Proof: See Appendix A.

Proposition 2 characterizes all of the SPE in the entry game with two symmetric firms, allowing for the possibility that payoffs are discontinuous. Lemma 6 outlines the equilibria in which there is rent equalization between the firms. There is a first-mover advantage in the SPE described in Lemma 7. Finally, Lemma 8 describes equilibria in which there is a second-mover advantage. As before the equilibria detailed in Lemmas 6, 7 and 8 are mutually exclusive. However, the LSPE, described by Lemma 5, is covered by one of Lemmas 6, 7 or 8.

This proposition generalizes existing models to cover discontinuous payoff functions. For example, Fudenberg and Tirole (1985, Section 5) discuss the possibility of discontinuities in the oligopoly case with three or more entrants. Similarly, Hoppe and Lehmann-Grube (2005) considers a possible discontinuity in the $L(t)$ payoff function (when $F(t)$ is monotonically decreasing). Encompassing these previous papers, we characterize all SPE for any finite number of discontinuities in both the leader's and the follower's payoff functions.

Consider, now, the possibility that one of the equilibria is an SSPE. The following corollary generalizes Corollary 1 to the case when payoffs are discontinuous.

Corollary 3. *If the equilibria can be ranked in the discontinuous game, the SSPE is the LSPE. A sufficient condition for the SSPE to exist is that: (i) there are no second-mover advantage equilibria; or (ii) there is a unique second-mover advantage equilibrium that is also the LSPE.*

Proof: Follows from the discussion above.

As in the continuous case with Corollary 1, when Corollary 3 holds, if there is no second-mover advantage equilibrium the SSPE is the SPE with the latest entry time. If a unique second-mover advantage equilibrium exists, this equilibrium is the SSPE.

Note that the techniques developed here can also be applied when $F(t)$ and $L(t)$ have discontinuities at the same time. This is a conceivable scenario, given the sort of event that produces a discontinuity – such as entry or exit in a related market – will potentially affect both the leader's and the follower's payoff. The ability to be able to handle joint discontinuities demonstrates both the generality and the usefulness of the solution algorithm outlined in Proposition 2.

Finally, it is worth noting that the existence of discontinuities generate additional SPE only if: (i) $C(0)$ contains points other than $t = 0$; (ii) a second-mover advantage equilibrium occurs at a discontinuity; or (iii) in the case that both (i) and (ii) hold. Practically, this means that often discontinuities have no real economic impact on the entry decision of firms. If this is the case the uniqueness results outlined in Section 3 continue to hold.

5 Concluding comments

The decision when to launch a new product is a critical question for many firms; it can determine profit, firm survival and the shape of markets. More generally, it drives economic development. Given its importance, innovation has received a great deal of attention from economists. We follow in this tradition by studying a market-entry game with complete information, when firm's actions are observable to all and there is no uncertainty.

We characterize all of the pure strategy subgame perfect equilibria for a two-player innovation game when the payoffs can potentially be non-monotonic, multiple-peaked and discontinuous. This new method is relevant in a variety of economic situations; for example, our algorithm can be applied to a product-innovation game with switching costs, to process innovation when there is an experience good, and to the timing of the sale of an asset. There can be non-standard payoffs in each of these examples, making them beyond the scope of existing techniques.

Our solution method allows us to distinguish between different types of equilibria in this general framework. We provide sufficient conditions that ensure: (i) equilibria can be Pareto ranked; and (ii) the equilibrium is unique.

6 Appendix A

Proof of Lemma 1

This proof consists of four parts: A, B, C and D. In Part A we show that all SPE with positive entry times must belong to $A(0)$. In Part B we prove that there exists a unique t^*, given by (2), at which either $L(t)$ is maximized over $A(0)$ or $t^* = 0$ when $A(0) = \emptyset$. Part C shows that t^* delivers the highest possible equilibrium payoff to the leader. Part D proves that t^* is an SPE.

(A) As a preliminary step, let us prove all SPE with entry time $t^* > 0$ must belong to $A(0)$. Assume, on the contrary, that there is an SPE with a positive entry time $t^* \notin A(0)$. It must be the case that either the condition $L(t) > L(\tau),\ \forall\ \tau \in [0, t^*)$, or the condition $F(t^*) \geq L(t^*)$ is not satisfied. If for some $\tau < t^*$ it is the case that $L(\tau) \geq L(t^*)$, the leader will have an incentive to enter earlier at τ. On the other hand, if $F(t^*) < L(t^*)$, the follower will have an incentive to preempt the leader and enter slightly earlier, as in Fudenberg and Tirole (1985). Neither of these situations are possible in equilibrium. Consequently, there is a contradiction, proving the statement that all SPE with positive entry times must belong to $A(0)$.

(B) Next, let us prove that there exists a unique t^* at which either $L(t)$ is maximized over $A(0)$ or $t^* = 0$ when $A(0) = \emptyset$. Specifically, t^* is given by

$$t^* = \begin{cases} \arg\max_t A(0) & \text{when} \quad A(0) \neq \emptyset, \\ 0 & \text{when} \quad A(0) = \emptyset. \end{cases} \tag{2}$$

When $A(0) = \emptyset$ the leader's optimal entry time is $t^* = 0$; we show this is part of an equilibrium strategy in Part D. Here we consider the situation when $A(0)$ is not empty.

Let us prove the existence of the solution to this problem of maximizing $L(t)$ over $A(0)$ when $A(0) \neq \emptyset$. Note that set $A(0)$ is bounded because t^{max} is finite, where t^{max} is the time t at which $L(t)$ reaches its global maximum (Assumption 4). We need to show that set $A(0)$ always contains its supremum. Assume that it does not. This means that there is a sequence $\{t_k\}$ contained in $A(0)$ that converges to some limit t^* that is not contained in set $A(0)$. This requires that either: there is $t' < t^*$ such that $L(t') \geq L(t^*)$; or that $F(t^*) < L(t^*)$. On the other hand, because sequence $\{t_k\}$ belongs to $A(0)$ it means that any $\tau \in [t', t^*)$ belongs to $A(0)$. Consequently, $L(\tau) > L(t')$ and $F(\tau) \geq L(\tau)$. This leads to a contradiction given that $L(t)$ and $F(t)$ are continuous functions, proving existence.

The uniqueness follows immediately from the way set $A(0)$ is constructed. If two entry times were to maximize $L(t)$ over $A(0)$, then the latter time would not belong to $A(0)$.

Next, let us show that if $t^* = \arg\max_{t \in A(0)} L(t)$, it is also the case that $t^* = \arg\max_t A(0)$ when $A(0) \neq \emptyset$. Assume the opposite that $t^* \neq \arg\max_t A(0)$. If $t^* < \arg\max_t A(0)$, then t^* does not maximize the leader's payoff over $A(0)$. If $t^* > \arg\max_t A(0)$, t^* does not belong to $A(0)$. Both situations lead to a contradiction. We have now shown that $t^* = \arg\max_t A(0)$, concluding the proof of Part B.

(C) Next, we prove that t^* given by (2) delivers the highest possible payoff to the leader. Given that in Part A we proved that all SPE with positive entry times must belong to $A(0)$, this point follows immediately.

(D) Let us prove that the proposed equilibrium with t^* defined in (2) is an SPE. When $A(0) \neq \emptyset$ there are three cases to consider for possible profitable deviations.

(1) If $L(t^*) = F(t^*)$, the strategies specified in the Lemma result in both firms entering at t^*, generating a payoff of $(L(t^*) + F(t^*))/2 = L(t^*)$ for both firms. If either of the firms enters earlier at $\tau < t^*$, that firm will get a payoff of $L(\tau)$. From the construction of set $A(0)$ in (1) it follows that $L(\tau) < L(t^*)$. On the other hand, if either firm enters later, that firm will get a payoff of $F(t^*)$, which is equal to $(L(t^*) + F(t^*))/2$. Consequently, if $L(t^*) = F(t^*)$ there is no profitable deviation for either firm.

(2) If $L(t^*) < F(t^*)$, one needs to consider deviations of the two firms separately. Without loss of generality, the first firm is the leader; it enters at t^* and gets a payoff of $L(t^*)$. The second firm is the follower; it gets a payoff of $F(t^*)$. If the follower deviates by entering earlier at some time $\tau < t^*$, it will get a payoff of $L(\tau) < L(t^*) < F(t^*)$. If it deviates by entering at t^*, it will get a payoff of $(L(t^*) + F(t^*))/2$, which is less than $F(t^*)$. If the follower enters at $t > t^*$, there will be no change to the equilibrium outcome (in terms of payoffs). Consequently, there is no profitable deviation for the follower.

If the leader deviates by entering earlier at some time $\tau < t^*$, it will get a payoff of $L(\tau) < L(t^*)$. If the leader deviates by entering later, it will get a smaller payoff because, as previously proved, t^* given by (2) delivers the highest possible payoff to the leader; see Part C of the proof.

(3) If $L(t^*) > F(t^*)$, an equilibrium with the leader entering at a positive time is not feasible. If this were the case, each firm would have an incentive to enter slightly earlier; consequently, the only possible equilibrium involves leader entry at $t^* = 0$. Note that in this case $\{0\} \notin A(0)$, meaning that $A(0) = \emptyset$.

Finally, let us consider the general case with $A(0) = \emptyset$. If $A(0) = \emptyset$, $L(0) > F(0)$. The strategies specified in the lemma result in both firms entering at $t^* = 0$. This generates a payoff of $(L(0) + F(0))/2$ for both firms. If either firm decides to enter later; it will get a payoff of $F(0)$, which is less than $(L(0) + F(0))/2$. Consequently, there is no profitable deviation for either firm and $t^* = 0$ is a unique SPE. This proves Part D, and concludes the proof of the Lemma. □

Proof of Lemma 2

Let us prove that if $x^* \in B$, x^* is an SPE. Given $L(t^*) = F(t^*)$, the strategies specified in the Lemma result in both firms entering at t^*, generating a payoff of $(L(t^*) + F(t^*))/2 = L(t^*)$ for both firms. There are two cases to consider for possible profitable deviations. If either firm enters earlier at $\tau < t^*$ it will get a payoff of $L(\tau)$. From the definition of set B in (5), it follows that $L(\tau) < L(t^*)$. On the other hand, if either firms enters later it will get a payoff of $F(t^*)$, which is equal to $(L(t^*) + F(t^*))/2$. Consequently, there is no profitable deviation for either firm if $L(t^*) = F(t^*)$. This proves the Lemma. □

Proof of Lemma 3

Let us prove that if $L(0) > F(0)$, $t^* = 0$ is the leader's entry time in the SPE. Given $L(0) > F(0)$, the strategies specified in the Lemma result in both firms entering at $t^* = 0$. This generates a payoff of $(L(0) + F(0))/2$ for both firms. If either firm decides to enter later it will get a payoff of $F(0)$, which is less than $(L(0) + F(0))/2$. Consequently, there is no profitable deviation for either firm from entering at $t^* = 0$ if $L(0) > F(0)$. The Lemma is proved. □

Proof of Lemma 4

Let us prove that if $t^* \in A(0)$, $t^* \notin B$ and the equilibrium is the LSPE of a given region U_s, this equilibrium is a second-mover advantage equilibrium of the entire game.

First, note that all possible equilibria with RE, when $L(t^*) = F(t^*)$, are covered by set B. Similarly, by construction of regions U_s, we do not need to consider possible first-mover advantage equilibrium with $L(0) > F(0)$. The only other possibility not already covered is when $F(t^*) > L(t^*)$.

Second, given $F(t^*) > L(t^*)$, one needs to consider deviations of the two firms separately. Without loss of generality, the first firm is the leader; it enters at t^* and gets a payoff of $L(t^*)$. The second firm is the follower; it gets a payoff of $F(t^*)$. Given $t^* \in A(0)$, if the follower deviates by entering at some time $\tau < t^*$, it will get a payoff of $L(\tau) < L(t^*) < F(t^*)$. If it deviates by entering at t^*, it will get a payoff of $(L(t^*) + F(t^*))/2$, which is less than $F(t^*)$. If the follower enters at $t > t^*$, there will be no change to the equilibrium outcome. Consequently, there is no profitable deviation for the follower.

Third, given $t^* \in A(0)$, if the leader deviates by entering earlier at some time $\tau < t^*$, it will get a payoff of $L(\tau) < L(t^*)$. If the leader deviates by entering later, it will get a smaller payoff because entering at t^* occurs in the LSPE for a given region U_s. Moreover, the strategies of both firms to enter whenever $L(t) \geq F(t)$ for $t > t^*$ ensure that entry cannot be postponed until after region U_s. This completes the proof. □

Proof of Proposition 1

Let us prove that there is no other SPE with the leader entering at t^*, that is not characterized in Lemmas 2, 3 and 4. There are three cases to consider.
(1) The case $L(t^*) = F(t^*)$ is covered by Lemma 2. Both firms entering at t^* is an SPE only if and only if condition $L(t^*) > L(\tau)\ \forall\ \tau \in [0, t^*)$ is satisfied. Otherwise firms will have an incentive to deviate by entering earlier.
(2) The case $L(t^*) > F(t^*)$ is covered by Lemma 3. Both firms entering at $t^* = 0$ is an SPE if and only if condition $L(0) > F(0)$ is satisfied. No other equilibria are possible in this case because in any candidate equilibrium with positive entry time both firms will have an incentive to deviate by entering earlier.
(3) The case with $L(t^*) < F(t^*)$ is covered by Lemma 4. The equilibrium is a second-mover advantage SPE if and only if: i) $t^* \in A(0)$; ii) $t^* \notin B$; and iii) the equilibrium is the LSPE of a given region U_s. If $t^* \notin A(0)$, firms will have an incentive to deviate by entering earlier. If the equilibrium is not the LSPE of a given region U_s the leader will have an incentive to enter at a different time. Furthermore, the condition that $t^* \notin B$ guarantees that it is a second-mover advantage SPE. This proves the proposition. □

Proof of Lemma 5

In a similar manner to the proof of Lemma 1, this proof consists of four parts: A, B, C and D. In Part A we show that all SPE must belong to $A(0) \cup C(0)$. In Part B we prove that there exists a unique t^* given by (13), at which $L(t)$ is maximized over $A(0) \cup C(0)$. Part C shows that t^* delivers the highest possible equilibrium payoff to the leader, while Part D proves that t^* is an SPE.

(A) As a preliminary step, let us prove all SPE must belong to $A(0) \cup C(0)$. Assume, on the contrary, that there is an SPE with entry time $t^* \notin [A(0) \cup C(0)]$. This requires us to consider two possible situations, one in which the candidate entry time occurs when payoffs are continuous and, second, when entry occurs at a point of discontinuity. In the case of continuous payoffs we apply the same arguments as outlined in the proof of Lemma 1. In the case of entry at a point of discontinuity, there two scenarios to consider. First, if $L(t^*) \leq F(t^*)$, the condition that $L(t) > L(\tau),\ \forall\ \tau \in [0, t^*)$ must hold, otherwise the leader will prefer to enter earlier at τ. This means that $t^* \in A(0)$. Second, if $L(t^*) > F(t^*)$ then $(L(t) + F(t))/2 >$

$L(\tau)\ \forall\ \tau \in [0,t)$ must hold to rule out possible preemption by the leader. This means that $t^* \in C(0)$. Consequently, t^* must belong to either $A(0)$ or $C(0)$.

(B) Next, we prove that there exists a unique t^* at which $L(t)$ is maximized over $A(0) \cup C(0)$, given by

$$t^* = \arg\max_t [A(0) \cup C(0)]. \tag{13}$$

Let us prove existence of the solution to this problem of maximizing $L(t)$ over $A(0) \cup C(0)$. Note that set $A(0)$ is bounded because t^{max} is finite, where t^{max} is the time t at which $L(t)$ reaches its global maximum (Assumption 4). In addition, set $C(0)$ is both closed and bounded because there is a finite number of discontinuities. If the supremum occurs at a point at which the payoffs are continuous, the arguments in Lemma 1 apply to show that set $A(0)$ always contains its supremum. When the supremum occurs at a point of discontinuity, we make use of Assumption 6 so as to ensure that set $A(0)$ contains its supremum. This proves existence.

Uniqueness follows immediately from the way set $A(0) \cup C(0)$ is constructed. If two points were to maximize $L(t)$ over $A(0) \cup C(0)$ then the one with the later time would not belong to $A(0) \cup C(0)$.

Next, using the same arguments as presented in the proof of Lemma 1, $t^* = \arg\max_{t \in A(0) \cup C(0)} L(t) = \arg\max_t [A(0) \cup C(0)]$. This concludes the proof of Part B.

(C) Let us prove that t^* given by (13) delivers the highest possible payoff to the leader. Given that in Part A we proved that all SPE must belong to $A(0) \cup C(0)$, this point follows immediately.

(D) Now we show that the proposed equilibrium with t^* defined in (13) is an SPE. When $L(t^*) = F(t^*)$ and $L(t^*) < F(t^*)$, the same arguments utilized in the proof of Lemma 1 apply. Consequently, let us concentrate on the case when $L(t^*) > F(t^*)$. If this is true, an equilibrium with the leader entering at a positive time at which the payoffs are continuous is not feasible because each firm would have an incentive to enter slightly earlier. As a result, the only possible equilibrium involves joint entry at $t^* = 0$ or at points of discontinuity. Note that as $t^* \in C(0)$, entering at t^* generates a payoff of $(L(t^*) + F(t^*))/2$ for both firms, which dominates any payoff from entering earlier. Entering later leads to a payoff of $F(t^*)$, which is less than $(L(t^*) + F(t^*))/2$. Consequently, there is no profitable deviation for either firm and t^* is an SPE. This proves Part D, and concludes the proof of the Lemma. □

Proof of Lemma 6

The same argument can be applied as in the case of Lemma 2. Note that the fact that there are discontinuities does not affect the argument. The Lemma therefore is proved. □

Proof of Lemma 7

Let us prove that if $t^* \in C(0)$, t^* is an SPE. With $L(t^*) > F(t^*)$, the strategies specified in the Lemma result in both firms entering at t^*, generating a payoff of

$(L(t^*) + F(t^*))/2$ for both firms. If either firm enters earlier at $\tau < t^*$ it will get a payoff of $L(\tau)$. From the definition of set $C(0)$ in (12) it follows that $L(\tau) < L(t^*)$. On the other hand, if either firms decides to enter later it will get a payoff of $F(t^*)$, which is less than $(L(t^*) + F(t^*))/2$. Consequently, there is no profitable deviation for either firm if $L(t^*) > F(t^*)$. The Lemma therefore is proved. □

Proof of Lemma 8

Let us prove that if $t^* \in A(0)$, $t^* \notin B$ and the equilibrium is the LSPE of a given region U_s, that equilibrium is a second-mover advantage equilibrium of the entire game.

Note that all possible equilibria with RE, when $L(t^*) = F(t^*)$, are covered by set B. Similarly, by construction of regions U_s, we do not need to consider possible first-mover advantage equilibrium with $L(t^*) > F(t^*)$. The only other possibility not already covered is when $F(t^*) > L(t^*)$.

The remaining arguments in Lemma 4, that there are no profitable deviations, apply in the case here despite the presence of discontinuities. This completes the proof. □

Proof of Proposition 2

Let us prove that there is no other SPE with the leader entering at t^* not characterized in Lemmas 6, 7 and 8. The proof closely follows the arguments made in the proof of Proposition 1. There are three cases to consider.

(1) The case $L(t^*) = F(t^*)$ is covered by Lemma 6. Both firms entering at t^* is an SPE if and only if condition $L(t^*) > L(\tau)\ \forall\ \tau \in [0, t^*)$ is satisfied. Otherwise firms will have an incentive to deviate by entering earlier.

(2) Lemma 7 focuses on the case when $L(t^*) > F(t^*)$. Both firms entering at $t^* \in C(0)$ is an SPE if and only if conditions $L(t^*) > F(t^*)$ and $(L(t) + F(t))/2 > L(\tau)\ \forall\ \tau \in [0, t)$ are satisfied. No other equilibria are possible in this case, because both firms will have an incentive to deviate by entering earlier in any candidate equilibrium that involves entry at a time other than points of discontinuity.

(3) Lemma 8 covers the case when $L(t^*) < F(t^*)$. The equilibrium is a second-mover advantage SPE if and only if $t^* \in A(0)$, $t^* \notin B$ and the equilibrium is the LSPE of a given region U_s. If $t^* \notin A(0)$, firms will have an incentive to deviate by entering earlier. If the equilibrium is not the LSPE of a given region U_s the leader will have an incentive to enter at a different time. Furthermore, the condition that $t^* \notin B$ guarantees that it is a second-mover advantage SPE. This completes the proof. □

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