Markov-Switching Models with Evolving Regime-Specific Parameters: Are Postwar Booms or Recessions All Alike?

Yunjong Eo [1]
University of Sydney
and
Chang-Jin Kim [2]
University of Washington and Korea University

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Abstract

In this paper, we relax the assumption of constant regime-specific mean growth rates in Hamilton's (1989) two-state Markov-switching model of the business cycle. We introduce a random walk hierarchical prior for each regime-specific mean growth rate and impose a cointegrating relationship between the mean growth rates in recessionary and expansionary periods. By applying the proposed model to postwar U.S. real GDP growth (1947:Q4-2011:Q3), we uncover the evolving nature of the regime-specific mean growth rates of real output in the U.S. business cycle. Additional features of the postwar U.S. business cycle that we uncover include: i) a steady decline in the long-run mean growth rate of real output over the postwar sample and ii) an asymmetric error-correction mechanism when the economy deviates from its long-run equilibrium.

Key Words: Business Cycle, Evolving Regime-Specific Parameters, Hierarchical Prior, Markov Switching, Error-Correction Dynamics, MCMC, State-Space Model

[1] **Eo:** School of Economics, University of Sydney, Sydney, NSW 2006, Australia [e-mail: yunjong.eo@sydney.edu.au]
[2] **Kim:** (Corresponding Author) Department of Economics, University of Washington, Seattle, WA. 98195, U.S.A. [e-mail: changjin@uw.edu]

1. Introduction

Blanchard and Watson (1986) raised an interesting question of whether or not business cycles are all alike. Their answer was "No." To motivate this paper, we ask,"Are postwar booms or recessions all alike?" Our answer is tentatively "No." In Hamilton's (1989) two-state Markov-switching model of the business cycle, the mean growth rates of real GDP during different episodes of a specific regime (boom or recession) are assumed to be the same. Although this assumption may be a reasonable approximation for a specific sample period, we claim that it may be a poor approximation for the extended sample that covers the whole postwar period. This is confirmed by Figure 1, in which the quarterly growth rates of real GDP for the sample period 1947:Q4 to 2011:Q3 are plotted along with the mean growth rate for each episode of NBER boom or recession. The shaded areas refer to the NBER recession periods. The mean growth rates for the 12 historical episodes of boom range from 0.59 to 1.83 with a standard deviation of 0.37. The mean growth rates for the 11 historical episodes of recession range from 0.02 to -0.69 with a standard deviation of 0.23. [3]

In this paper, we propose a flexible two-state Markov-switching model of the business cycle, in which the regime-specific mean growth rates of real output may evolve over different episodes of boom or recession. We first present a preliminary model, in which we assume a simple random walk hierarchical prior for each regime-specific mean growth rate. Within this framework, we provide basic insights into the model. Then, by imposing a condition for the existence of the long-run mean growth rate for real output, we extend the preliminary model to a realistic one, in which we allow for a cointegrating relationship between the two regime-specific mean growth rates.

For making an inference about the model, we build on recent advances in Bayesian approaches to change-point models that allow for flexible relationships between parameters in various regimes and/or an unknown number of structural breaks. (Koop and Potter (2007), Giordani and Kohn (2008), Geweke and Jiang (2011), etc.) In particular, we follow Koop and Potter (2007) and cast the models into standard Markov-switching state-space formulations with heteroscedastic shocks to regime-specific parameters. Once the model is put into

[3] For more details, the summary statistics are provided in Appendix Table A.1.

standard state-space formulations, a Markov Chain Monte Carlo (MCMC) procedure can be easily developed based on the existing posterior simulation method for state-space models and on that for Markov-switching models. For example, in order to generate the evolving regime-specific mean growth rates conditional on the Markov-switching regime indicator variable, we can take advantage of Carter and Kohn's (1994) and Kim et al.'s (1998) methods of posterior simulation for linear state-space models. In order to generate the Markov-switching regime indicator variable conditional on the evolving regime-specific mean growth rates, we use a modified version of Albert and Chib's (1993) method.

We estimate the proposed model and various competing models using postwar U.S. real GDP growth for the sample period of 1947:Q4 to 2011:Q3. In our empirical models, we also allow for the possibility of time-varying long-run mean growth rate and stochastic volatility for the disturbance term. The performance of the proposed model is superior to that of various other competing models, including the Hamilton (1989) model and the Hamilton model with the 'bounce-back effect' of Kim et al. (2005), both in identifying recessions and in making inferences about the mean growth rates. The superiority of the proposed model is also confirmed by Bayesian model comparison based on the Deviance Information Criterion (DIC).

The evolving nature of each regime-specific mean growth rate for booms or recessions is not the only feature of the U.S. postwar business cycle that we uncover in this paper. First, we find that the decline in the long-run mean growth rate of real output was not abrupt. Whereas an abrupt decline in the long-run output growth has commonly been postulated around the mid-1970s (e.g., Perron (1989) and Zivot and Andrews (2002)), we find that the long-run mean growth rate has declined steadily over the entire postwar sample, as first documented by Stock and Watson (2012). Second, an asymmetric error-correction effect arises when the economy deviates from its long-run equilibrium. We find more evidence of error correction in the form of a high growth expansion following a deep recession than vice versa.

The remainder of this paper is organized as follows. In Section 2, we propose a model with evolving regime-specific mean growth rates of real output, in which we impose a condition for the existence of the long-run mean growth rate. In Section 3, we present a state-space representation of the proposed model and describe an MCMC procedure for Bayesian inference of the model. In Section 4, we apply the proposed model to postwar U.S. real GDP growth. The performance of the proposed model is compared to those of various alternative models. Section 5 provides the summary and conclusions.

2. Markov-Switching Models with Evolving Regime-Specific Mean Growth Rates

2.1. A Preliminary Model

Let y_t be real output growth, and consider the following Markov-switching model of the business cycle:

$$y_t = (1 - S_t)\bar{\mu}_{0,\tau_0} + S_t\bar{\mu}_{1,\tau_1} + e_t, \quad S_t = 0, 1, \tag{1}$$

$$e_t \sim i.i.d.N(0, \sigma_e^2), \tag{2}$$

$$t = 1, 2, ..., T; \quad \tau_0 = 1, 2, ..., N_0; \quad \tau_1 = 1, 2, ..., N_1,$$

where $\bar{\mu}_{0,\tau_0}$ is the mean growth rate during the $\tau_0 - th$ episode of boom in the sample; $\bar{\mu}_{1,\tau_1}$ is the mean growth rate during the $\tau_1 - th$ episode of recession; and N_0 and N_1 are the total numbers of the episodes of boom and recession, respectively, conditional on the states. Note that N_0 and N_1 are random variables and that they are dependent upon the realizations of the latent state variables $\tilde{S}_T = [\, S_1 \quad S_2 \quad \dots \quad S_T\,]'$ that characterize the business cycle regime. The latent state variable S_t follows a first-order Markov-switching process with the transition probabilities:

$$Pr[S_t = 1|S_{t-1} = 1] = p, \quad Pr[S_t = 0|S_{t-1} = 0] = q. \tag{3}$$

While Hamilton (1989) assumes that $\bar{\mu}_{0,\tau_0} = \mu_0$ for all $\tau_0 = 1, 2, ..., N_0$ and $\bar{\mu}_{1,\tau_1} = \mu_1$ for all $\tau_1 = 1, 2, ..., N_1$, we allow for the possibility that different episodes of boom (or recession) have different mean growth rates. In order to allow for the dependence of mean growth rates between current and past episodes of boom or recession, we adopt hierarchical priors given by the following random walk dynamics for $\bar{\mu}_{0,\tau_0}$ and $\bar{\mu}_{1,\tau_1}$:

$$\bar{\mu}_{0,\tau_0} = \bar{\mu}_{0,\tau_0-1} + \omega_{0,\tau_0}, \quad \omega_{0,\tau_0} \sim i.i.d.N(0, \sigma^2_{\omega,0}), \tag{4}$$

$$\bar{\mu}_{1,\tau_1} = \bar{\mu}_{1,\tau_1-1} + \omega_{1,\tau_1}, \quad \omega_{1,\tau_1} \sim i.i.d.N(0, \sigma^2_{\omega,1}), \tag{5}$$

$$\tau_0 = 1, 2, ..., N_0; \qquad \tau_1 = 1, 2, ..., N_1,$$

where ω_{0,τ_0} and ω_{1,τ_1} are independent of each other and are not correlated with e_t in equation (2). Within the context of the linear models with multiple structural breaks, Koop and Potter (2007) use the same hierarchical prior in order to allow for dependence in parameters across regimes. When $\sigma^2_{\omega,0} = \sigma^2_{\omega,1} = 0$ the above model collapses to that of Hamilton (1989). The fundamental difference between the model proposed in this paper and the Hamilton model is illustrated in Figure 2.

For notational simplicity, we rewrite equations (4) and (5) with a common subscript τ instead of two different regime subscripts τ_0 and τ_1, in the following way:

$$y_t = (1 - S_t)\bar{\mu}_{0,\tau} + S_t\bar{\mu}_{1,\tau} + e_t, \quad S_t = 0, 1; \quad t = 1, 2, ..., T, \tag{6}$$

$$\bar{\mu}_{0,\tau} = \bar{\mu}_{0,\tau-1} + \omega_{0,\tau}, \quad \omega_{0,\tau} \sim N(0, \sigma^2_{0,\tau}), \tag{7}$$

$$\bar{\mu}_{1,\tau} = \bar{\mu}_{1,\tau-1} + \omega_{1,\tau}, \quad \omega_{1,\tau} \sim N(0, \sigma^2_{1,\tau}), \tag{8}$$

$$\sigma^2_{0,\tau} = \begin{cases} \sigma^2_{\omega,0}, & \text{if } \ \tau = (2j-1) + j^*, \ j = 1, 2, 3, \ldots \\ 0, & \text{otherwise,} \end{cases} \tag{9}$$

$$\sigma^2_{1,\tau} = \begin{cases} \sigma^2_{\omega,1}, & \text{if } \ \tau = 2j - j^*, \ j = 1, 2, 3, \ldots \\ 0, & \text{otherwise,} \end{cases} \tag{10}$$

where $j^* = 1$, if the sample starts with the first episode of recession; and $j^* = 0$, if the sample starts with the first episode of boom. We therefore have $\tau = 1, 2, \ldots, N$, where $N = N_1 + N_2$.

2.2. A Realistic Model with a Long-Run Restriction: Vector Error-Correction Dynamics for Regime-Specific Mean Growth Rates

One weakness of the preliminary model in Section 2.1 is that the long-run growth rate does not exist. This results in a serious problem especially when the assumption of a constant long-run growth rate is relaxed in a later section. In this section, we first derive a condition for the existence of the long-run growth rate.

By denoting the long-run growth rate as δ, we rewrite equation (6) as:

$$y_t = \delta + (1 - S_t)\mu_{0,\tau} + S_t\mu_{1,\tau} + e_t, \tag{11}$$

$$e_t \sim i.i.d.N(0, \sigma_e^2), \tag{12}$$

where $\mu_{0,\tau}$ or $\mu_{1,\tau}$ refers to the deviation of the mean growth rate during boom or recession from the long-run mean growth rate δ.

By taking unconditional expectations on both sides of equation (11), we obtain the restriction: $E[y_t - \delta] = E[(1 - S_t)\mu_{0,\tau} + S_t\mu_{1,\tau} + e_t] = 0$, which, due to the law of iterated expectations and the independence of S_t from $\mu_{0,\tau}$ or $\mu_{1,\tau}$ for all t and τ, results in:

$$E[\pi_0\mu_{0,\tau} + \pi_1\mu_{1,\tau}] = 0 \tag{13}$$

where, $\pi_i = Pr[S_t = i]$, $i = 0, 1$, are the unconditional probabilities of boom ($i = 0$) and recession ($i = 1$).

This long-run restriction, combined with the random walk assumptions for the regime-specific mean growth rates, suggests that $\mu_{0,\tau}$ and $\mu_{1,\tau}$ are cointegrated with a cointegrating vector $[\,\pi_0 \quad \pi_1\,]'$. Thus, changes in $\mu_{0,\tau}$ and $\mu_{1,\tau}$ do not have any long-run effect on y_t by the long-run restriction of the cointegration. The following describes the dynamics of $\mu_{0,\tau}$ and $\mu_{1,\tau}$ with the long-run restriction:

$$\mu_{0,\tau} = \mu_{0,\tau-1} + \theta_0(\pi_0\mu_{0,\tau-1} + \pi_1\mu_{1,\tau-1}) + \omega_{0,\tau}, \quad \omega_{0,\tau} \sim N(0, \sigma_{0,\tau}^2), \tag{14}$$

$$\mu_{1,\tau} = \mu_{1,\tau-1} + \theta_1(\pi_0\mu_{0,\tau-1} + \pi_1\mu_{1,\tau-1}) + \omega_{1,\tau}, \quad \omega_{1,\tau} \sim N(0, \sigma_{1\tau}^2), \tag{15}$$

$$\mu_{0,\tau} > 0 \;\; and \;\; \mu_{1,\tau} < 0, \;\; \forall \;\; \tau,$$

$$\tau = 1, 2, ..., N,$$

where $\sigma_{0\tau}^2$ and $\sigma_{1\tau}^2$ are defined in (9) and (10).

Finally, in order to guarantee the stability of the above vector error-correction model and the existence of long-run output growth, we actually need a restriction on error-correction coefficients θ_0 and θ_1. If we cast the vector error-correction model in equations (14) and (15) into a state-space form, we have:

$$\begin{bmatrix} \Delta\mu_{0,\tau} \\ \Delta\mu_{1,\tau} \\ z_\tau \end{bmatrix} = \begin{bmatrix} 0 & 0 & \theta_0 \\ 0 & 0 & \theta_1 \\ 0 & 0 & 1+\theta_0\pi_0+\theta_1\pi_1 \end{bmatrix} \begin{bmatrix} \Delta\mu_{0,\tau-1} \\ \Delta\mu_{1,\tau-1} \\ z_{\tau-1} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \pi_0 & \pi_1 \end{bmatrix} \begin{bmatrix} \omega_{0,\tau} \\ \omega_{1,\tau} \end{bmatrix}, \tag{16}$$

$$\begin{bmatrix} \omega_{0,\tau} \\ \omega_{1,\tau} \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma^2_{\omega,0} & 0 \\ 0 & \sigma^2_{\omega,1} \end{bmatrix} \right), \tag{17}$$

$$\tau = 1, 2, ..., N,$$

where $z_\tau = \pi_0\mu_{0,\tau} + \pi_1\mu_{1,\tau}$ is the equilibrium error during period τ. [4] Here, as the equilibrium error needs to be stationary, the restriction on the θ_0 and θ_1 parameters is given by:

$$-1 < 1 + \theta_0\pi_0 + \theta_1\pi_1 < 1. \tag{18}$$

The model in equations (11)-(15) differs from a conventional unobserved-components model. The regime-specific mean growth rate $\mu_{0,\tau}$ or $\mu_{1,\tau}$ changes only when we face a new episode of boom or recession. Furthermore, adjustment to long-run equilibrium occurs only when the regime changes from boom to recession or vice versa. Thus, we can cast the model into the following unobserved-components representation of the model:

<u>*Conventional Unobserved-Components Model Representation*</u>

$$y_t = \delta + (1 - S_t)\mu^*_{0,t} + S_t\mu^*_{1,t} + e_t, \tag{19}$$

$$\mu^*_{0,t} = \mu^*_{0,t-1} + \theta_0(d_{10,t} + d_{01,t})(\pi_0\mu^*_{0,t-1} + \pi_1\mu^*_{1,t-1}) + \omega^*_{0,t}, \quad \omega^*_{0,t} \sim N(0, d_{10,t}\sigma^2_{\omega,0}), \tag{20}$$

$$\mu^*_{1,t} = \mu^*_{1,t-1} + \theta_1(d_{10,t} + d_{01,t})(\pi_0\mu^*_{0,t-1} + \pi_1\mu^*_{1,t-1}) + \omega^*_{1,t}, \quad \omega^*_{1,t} \sim N(0, d_{01,t}\sigma^2_{\omega,1}), \tag{21}$$

$$\mu^*_{0,t} > 0 \;\; and \;\; \mu^*_{1,t} < 0, \quad \forall \;\; t.$$

$$-1 < 1 + \theta_0\pi_0 + \theta_1\pi_1 < 1,$$

[4] The long-run equilibrium error dynamics $z_t = (1+\theta_0\pi_0+\theta_1\pi_1)z_{t-1} + (\pi_0\omega_{0,\tau} + \pi_1\omega_{1,\tau})$ in (16) can be derived by multiplying (14) and (15) by π_0 and π_1, respectively, and summing them up.

$$t = 1, 2, ..., T,$$

where

$$d_{ij,t} = \begin{cases} 1, & \text{if } \ S_{t-1} = i \ and \ S_t = j, \quad j \neq i; \\ 0, & \text{otherwise.} \end{cases} \tag{22}$$

3. Markov Chain Monte Carlo (MCMC) Procedure and Model Comparison

3.1. Markov Chain Monte Carlo (MCMC) Procedure

As in Koop and Potter (2007), we first cast the unobserved components model derived in the previous section into a state-space model:

Measurement Equation

$$y_t = \delta + [\,(1 - S_t) \quad S_t\,] \begin{bmatrix} \mu^*_{0,t} \\ \mu^*_{1,t} \end{bmatrix} + e_t, \quad e_t \sim i.i.d.N(0, \sigma^2_e), \tag{23}$$

$$\left(\Leftrightarrow \quad y_t = \delta + H_t \mu^*_t + e_t, \quad e_t \sim i.i.d.N(0, \sigma^2_e)\right)$$

State Equation

$$\begin{bmatrix} \mu^*_{0,t} \\ \mu^*_{1,t} \end{bmatrix} = \begin{bmatrix} 1 + \theta_0 \pi_0 (d_{10,t} + d_{01,t}) & \theta_0 \pi_1 (d_{10,t} + d_{01,t}) \\ \theta_1 \pi_0 (d_{10,t} + d_{01,t}) & 1 + \theta_1 \pi_1 (d_{10,t} + d_{01,t}) \end{bmatrix} \begin{bmatrix} \mu^*_{0,t-1} \\ \mu^*_{1,t-1} \end{bmatrix} + \begin{bmatrix} \omega^*_{0,t} \\ \omega^*_{1,t} \end{bmatrix} \tag{24}$$

$$\left(\Leftrightarrow \quad \mu^*_t = F_t \mu^*_{t-1} + \omega_t, \quad \omega_t \sim N(0, \Omega_t)\right),$$

where $\Omega_t = Diag\,(\,d_{10,t}\sigma^2_{\omega,0}, \ d_{01,t}\sigma^2_{\omega,1}\,)$ and $d_{ij,t}$ is as defined in (22).

Conditional on $\tilde{S}_T = [\,S_1 \quad S_2 \quad \ldots \quad S_T\,]'$, the above is a linear state-space model with heteroscedastic shocks and a procedure for making inferences on $\mu^*_{0,t}$ and $\mu^*_{1,t}$ (the elements of the state vector μ^*_t) can easily be developed by modifying the procedure proposed by Carter and Kohn (1994). Furthermore, conditional on the $\mu^*_{0,t}$ and $\mu^*_{1,t}$ terms generated for $t = 1, 2, ..., T$, a procedure for generating the regime indicator variable S_t can be derived easily by modifying the procedure proposed by Albert and Chib (1993). In what follows, we summarize the prior used for Bayesian inference of the model and present an outline for the MCMC procedure.

By defining $\tilde{\mu}^*_{j,T} = [\,\mu^*_{j,1} \quad \mu^*_{j,2} \quad \ldots \quad \mu^*_{j,T}\,]'$, $j = 0, 1$ and $\tilde{S}_T = [\,S_1 \quad S_2 \quad \ldots \quad S_T\,]'$, the full specification for the priors can be summarized as:

<u>*Summary of the Prior*</u>

$$
\begin{aligned}
&p(\tilde{\mu}^*_{0,T},\tilde{\mu}^*_{1,T}, \tilde{S}_T, \mu^*_{0,0}, \mu^*_{1,0}, S_0, \delta, \sigma^2_e, \sigma^2_{\omega,0}, \sigma^2_{\omega,1}, \theta_0, \theta_1, p, q) \\
&\quad = p(\tilde{\mu}^*_{0,T}, \tilde{\mu}^*_{1,T}|\mu^*_{0,0}, \mu^*_{1,0}, \tilde{S}_T, S_0, \delta, \sigma^2_e, \sigma^2_{\omega,0}, \sigma^2_{\omega,1}, \theta_0, \theta_1) \times p(\tilde{S}_T|S_0, p, q) \\
&\qquad \times p(\mu^*_{0,0}, \mu^*_{1,0}, S_0, \delta, \sigma^2_e, \sigma^2_{\omega,0}, \sigma^2_{\omega,1}, \theta_0, \theta_1, p, q) \\
&\quad = \left[\prod_{t=1}^{T} p(\mu^*_{0,t}, \mu^*_{1,t}|\mu^*_{0,t-1}, \mu^*_{1,t-1}, S_t, S_{t-1}, \sigma^2_{\omega,0}, \sigma^2_{\omega,1}, \theta_0, \theta_1)\right] \\
&\qquad \times \left[\prod_{t=1}^{T} p(S_t|S_{t-1}, p, q)\right] \times p(\mu^*_{0,0}, \mu^*_{1,0}) \times p(S_0) \times p(\delta|\sigma^2_e) \times p(\sigma^2_e) \\
&\qquad \times p(\theta_0|\sigma^2_{\omega,0}) \times p(\sigma^2_{\omega,0}) \times p(\theta_1|\sigma^2_{\omega,1}) \times p(\sigma^2_{\omega,1}) \times p(p, q),
\end{aligned} \tag{25}
$$

where the joint conditional prior $p(\mu^*_{0,t}, \mu^*_{1,t}|\mu^*_{0,t-1}, \mu^*_{1,t-1}, S_t, S_{t-1}, \sigma^2_{\omega,0}, \sigma^2_{\omega,1}, \theta_0, \theta_1)$ is given by equation (24); $p(S_t|S_{t-1}, p, q)$ is given by the transition probabilities in (3); $p(\mu^*_{0,0}, \mu^*_{1,0})$ is diffuse; $p(S_0)$ is given by the unconditional probabilities of S_t; $p(\delta|\sigma^2_e)$, $p(\theta_0|\sigma^2_{\omega,0})$ and $p(\theta_1|\sigma^2_{\omega,1})$ are independent normals; $p(\sigma^2_e)$, $p(\sigma^2_{\omega,0})$, and $p(\sigma^2_{\omega,1})$ are independent inverted Gamma's; and $p(q, p)$ are independent Beta's.

<u>*Outline of the MCMC Procedure*</u>

<u>**Step 0:**</u>

Initialize the parameters of the model $\tilde{\psi} = [\,\delta \quad \sigma^2_e \quad \theta_0 \quad \sigma^2_{\omega,0} \quad \theta_1 \quad \sigma^2_{\omega,1} \quad q \quad p\,]'$ and the states $\tilde{S}_T = [\,S_1 \quad S_2 \quad \ldots \quad S_T\,]'$.

<u>**Step 1:**</u>

Generate $\tilde{\mu}^*_{0,T} = [\,\mu^*_{0,1} \quad \mu^*_{0,2} \quad \ldots \quad \mu^*_{0,T}\,]'$ and $\tilde{\mu}^*_{1,T} = [\,\mu^*_{1,1} \quad \mu^*_{1,2} \quad \ldots \quad \mu^*_{1,T}\,]'$ conditional on $\tilde{\psi}$, $\tilde{S}_T$, and data $\tilde{Y}_T = [\,y_1 \quad y_2 \quad \ldots \quad y_T\,]'$. This step is based on the state-space representation of the model in equations (23) and (24). The conditional joint posterior distribution of $\tilde{\mu}^*_{0,T}$ and $\tilde{\mu}^*_{1,T}$ can be decomposed as:

$$
p(\tilde{\mu}^*_{0,T}, \tilde{\mu}^*_{1,T}|\tilde{Y}_T, \tilde{S}_T, \tilde{\Psi}) = p(\mu^*_{0,T}, \mu^*_{1,T}|\tilde{Y}_T, \tilde{S}_T, \tilde{\psi}) \prod_{t=1}^{T-1} p(\mu^*_{0,t}, \mu^*_{1,t}|\mu^*_{0,t+1}, \mu^*_{1,t+1}, \tilde{Y}_t, \tilde{S}_T, \tilde{\psi}), \tag{26}
$$

which suggests that we can sequentially generate $\mu^*_{0,t}$ and $\mu^*_{1,t}$ for $t = T, T-1, \ldots, 2, 1$.

Step 2:

Generate $\tilde{S}_T$ conditional on $\tilde{\mu}^*_{0,T}$ and $\tilde{\mu}^*_{1,T}$; parameters $\tilde{\psi}$; and data $\tilde{Y}_T$. This step is based on equation (23) and the transition probabilities in (3). As in Albert and Chib (1993), $p(S_t|\tilde{Y}_T, \tilde{S}_{\neq t}, \tilde{\mu}^*_{0,T}, \tilde{\mu}^*_{1,T}, \tilde{\psi})$ can be derived as:

$$p(S_t|\tilde{Y}_T, \tilde{S}_{\neq t}, \tilde{\mu}^*_{0,T}, \tilde{\mu}^*_{1,T}, \tilde{\psi}) \propto Pr(S_t|S_{t-1})Pr(S_{t+1}|S_t)p(y_t|\tilde{Y}_{t-1}, S_t, \mu^*_{0,t}, \mu^*_{1,t}, \tilde{\psi}). \quad (27)$$

Step 3:

Generate θ_0, θ_1, $\sigma^2_{\omega,0}$ and $\sigma^2_{\omega_1}$, conditional on $\tilde{\mu}^*_{0,T}$, $\tilde{\mu}^*_{1,T}$, and $\tilde{S}_T$. This step is based on equations (14)-(15).

Step 4:

Generate δ and σ^2_e, conditional on $\tilde{\mu}^*_{0,T}$, $\tilde{\mu}^*_{1,T}$,, $\tilde{S}_T$ and $\tilde{Y}_T$. This step is based on equation (23).

Step 5: Generate q and p conditional on $\tilde{S}_T$.

For more details of the above MCMC procedure, readers are referred to Appendix A.

3.2. Model Selection Criterion

In addition to visually inspecting the estimated probabilities of being in the recession regime with the NBER recession dates, we formally compare our proposed model with various extensions of the Hamilton model. The usual method of Bayesian model comparison is through marginal likelihood calculations but they are quite sensitive to prior information especially for the models with high-dimensional parameter spaces. The proposed model has a complicated hierarchical structure for the evolving regime-specific mean growth rates. Furthermore, when the model is extended to incorporate a random walk process for the long-run mean growth and stochastic volatility for the disturbance terms, the hierarchical structural of the model is further complicated. Thus, the number of model parameters and the latent state variables is extremely large and easily exceeds the number of observations. This creates difficulty in evaluating the marginal likelihood.

In order to overcome the difficulty in marginalizing over the parameter vector and the latent state variables, we adopt the Deviance Information Criterion (DIC) for our model comparisons. Spiegelhalter et al. (2002) first proposed using DIC for complex hierarchical models and Berg, Meyer, and Yu (2004) showed that DIC can be effectively used for comparing various stochastic volatility models. As discussed, DIC is developed exactly for the models such as ours. This model selection criterion consists of (i) a Bayesian measure of model fit defined as the posterior expectations of the deviance and (ii) a penalty term to measure the complexity of the model as in the Akaike Information Criterion (AIC) and the Schwarz Information Criterion (SIC). The penalty term represents the effective number of parameters defined by the difference between the posterior mean of the deviance and the deviance evaluated at the posterior mean of the parameters [5]:

$$DIC = \underbrace{-2E_{\Psi|y}[\log f(y|\Psi)]}_{(i)model fit} + \underbrace{2\left\{\log f(y|\overline{\Psi}) - E_{\Psi|y}\left[\log f(y|\Psi)\right]\right\}}_{(ii)penalty} \tag{28}$$

where Ψ is a collection of model parameters including state variables and $\overline{\Psi}$ is its posterior mean. Thus, DIC prefers the model with a small value. In practice, the posterior expectations of the deviance are calculated with

$$E_{\Psi|y}\left[\log f(y|\Psi)\right] = \frac{1}{M}\sum_{j=1}^{M}\log f(y|\Psi_j) \tag{29}$$

where M is the number of MCMC simulations. Thus, calculating these two terms is easy when MCMC draws are readily available. We simply need to average the log of likelihoods from MCMC draws and evaluate the log of likelihood at the mean of MCMC draws for the parameters.

4. An Application to U.S. Real GDP Growth Data

4.1. The Hamilton Model and the Proposed Model

[5] For more details, refer to Spiegelhalter et al. (2002).

We apply the proposed model and the MCMC procedure presented in Section 3 to postwar U.S. real GDP growth data over the sample period from 1947:Q4 to 2011:Q3. The results are compared to those from the Hamilton (1989) model. For both models, we specify the long-run mean growth rate as a random walk process and the variance of the disturbance terms as a stochastic volatility process. The former is incorporated in order to reflect Stock and Watson's (2012) observation that long-run growth of real output has declined steadily over the postwar sample and the latter is incorporated to reflect the Great Moderation.

For the two competing models under consideration, the long-run growth rate (δ) in equation (11) is replaced by:

$$\delta_t = \delta_{t-1} + \epsilon_t, \quad \epsilon_t \sim i.i.d.N(0, \sigma_\epsilon^2), \tag{30}$$

and the distribution of the disturbance term e_t in equation (12) is replaced by:

$$e_t \sim N(0, \sigma_{e,t}^2), \tag{31}$$

where

$$ln(\sigma_{e,t}^2) = ln(\sigma_{e,t-1}^2) + \eta_t, \quad \eta_t \sim i.i.d.N(0, \sigma_\eta^2). \tag{32}$$

The two competing models with the features in equations (30)-(32) are summarized below:

Model I: Hamilton Model

$$y_t = \delta_t + (1 - S_t)\mu_0 + S_t\mu_1 + e_t, \quad e_t \sim N(0, \sigma_{e,t}^2),$$

$$\mu_0 > 0, \quad \mu_1 < 0,$$

$$\pi_0\mu_0 + \pi_1\mu_1 = 0,$$

where the regime-specific mean growth rates (μ_0 and μ_1) are assumed to be time-invariant; the transition probabilities for S_t are as given in equation (3); and the last equation above is the identifying restriction that we need for estimating the long-run growth rate δ_t. Here, π_0 and π_1 refer to steady-state probabilities of S_t.

Model II: Proposed Model

$$y_t = \delta_t + (1 - S_t)\mu_{0,\tau} + S_t\mu_{1,\tau} + e_t, \quad e_t \sim N(0, \sigma^2_{e,t}),$$

$$\mu_{0,\tau} > 0, \quad \mu_{1,\tau} < 0, \quad for\ all\ \tau,$$

$$E[\pi_0\mu_{0,\tau} + \pi_1\mu_{1,\tau}] = 0,$$

where the hierarchical prior for $\mu_{0,\tau}$ and $\mu_{1,\tau}$ are given in equations (14) and (15) along with equation (18), which restricts the coefficients governing the speed of adjustment in their error-correction dynamics; and the transition probabilities for S_t are the same as in the Hamilton model. The priors we specify for the variances of the shocks are Inverted Gamma distributions, those for the transition probabilities are Beta distributions; and those for all the other parameters are Normal distributions. The prior mean and the the standard deviations for the parameters common to both models are set to be the same.

All inferences are based on 20,000 Gibbs simulations after discarding 10,000 burn-ins. Tables 1 and 2 summarize prior and posterior moments for the Hamilton model and those for the proposed model, respectively. Of particular interest in Table 2 are the posterior distributions for the θ_0 and θ_1 coefficients in the proposed model. These coefficients represent the speed of adjustments at which the regime-specific mean growth rates converge to long-run equilibrium. Even though the posterior means for θ_0 an θ_1 are both negative, the sample evidence in favor of $\theta_0 < 0$ is relatively more than that in favor of $\theta_1 < 0$. [6] Notice that, while the upper bound for the 90% bands for θ_0 is close to 0, that for θ_1 is much greater than zero. This suggests that, on average, a relatively strong recovery would follow a severe recession, which is sometimes referred to as the 'bounce-back effect' in the literature (e.g., Beaudry and Koop (1993), Kim, Morley, and Piger (2005), and references there-in).

DIC for the proposed model (421.50) is considerably lower than that for the Hamilton model (446.79), suggesting that the proposed model is preferred to the Hamilton model. The plots of posterior regime probabilities and posterior mean growth rates for the two models further confirm this. In Figure 3, the posterior probabilities of recession for both models are depicted against the NBER recessions (shaded areas). The proposed model clearly does a better job in the in-sample prediction of the NBER recession than the Hamilton model.

[6] The results are robust with respect to alternative prior moments used for the θ_0 and θ_1 coefficients.

In Figure 4, the posterior mean growth rates from the two competing models are depicted against the NBER business cycle episode-specific mean growth rates. [7] Again, the proposed model does a much better job in replicating the NBER episode-specific mean growth rates.

Figure 5 compares the long-run mean growth rates. The gradual declines in the long-run mean growth rates throughout the sample and their magnitudes for both models are in close agreement with that reported in Stock and Watson (2012). [8] A slight difference is that, whereas the posterior long-run mean growth for the Hamilton model steadily declines from close to 1% to 0.4% throughout the sample, that for the proposed model declines from around 0.8% to 0.5%. Clearly, a much higher long-run mean growth rate in the 1950s and a bigger decline for the Hamilton model may be an artifact of assuming constant regime-specific means, as shown in Figure 4.

Lastly in Figure 6, we compare the volatility of the disturbance terms for the two models. The volatility before the 1980s for the Hamilton model is estimated to be about 25%∼50% higher than that for the proposed model. Again, this may be due to the constant regime-specific mean growth assumed in the Hamilton model, and a lot of the variation in the regime-specific mean growth rates during this period is reflected as high variance of the disturbance terms.

4.2. Robustness Check: Various Alternative Model Specifications

In this section, we consider and estimate various alternative models. For example, one important finding from the proposed model is the existence of high growth recoveries that typically follow deep recessions (i.e., the 'bounce-back effect' of Beaudry and Koop (1993), Kim, Morley, and Piger (2005), and references there-in). One cannot rule out the possibility that the estimated evolving regime-specific mean growth rates for the proposed model may

[7] The posterior mean growth rate is calculated from $E[y_t|\tilde{Y}_T]$ using Gibbs draws.

[8] Stock and Watson (2012) support this finding by examining various macroeconomic variables and suggest that the declining trend growth rate is due to changes in underlying demographic factors, especially (i) the stagnant female labor force participation rate and (ii) the aging of the U.S. workforce. Note that this nature of changes in the long-run output growth is quite different from what has been reported in the literature. The literature suggests an abrupt decline in the long-run mean growth rate after the first Oil Shock of the mid-1970s (e.g. Perron (1989) and Zivot and Andrews (2002)).

simply be an artifact of not explicitly considering the 'bounce-back effect' in the Hamilton model. We thus consider an extended Hamilton model, in which the 'bounce-back effect' is incorporated as in Kim et. al (2005):

Model III: Hamilton Model with a Bounce-Back Effect

$$y_t = \delta_t + (1 - S_t)\mu_0 + S_t\mu_1 + \lambda \sum_{j=1}^{m} S_{t-j}(y_{t-j} - \delta_{t-j}) + e_t, \quad e_t \sim N(0, \sigma_{e,t}^2),$$

$$\mu_0 > 0, \quad \mu_1 < 0,$$

$$\pi_0\mu_0 + \pi_1\mu_1 = 0,$$

where the specification for the transition probabilities, the long-run growth rate δ_t, and stochastic volatility $\sigma_{e,t}^2$ are the same as in Models I and II. A nice feature of the above model is that, the longer the duration and the deeper the magnitude of a recession, the higher recovery (i.e., a bigger bounce-back effect). The bounce-back effect term $\lambda \sum_{j=1}^{m} S_{t-j}(y_{t-j} - \delta_{t-j})$ reflects this feature. Note that when $\lambda = 0$ the above model collapses to Model I (the Hamilton model). We follow Kim, Morley, and Piger (2005), in setting $m = 6$ for the length of the bounce-back effect.

Table 3 presents prior and posterior moments. The prior distribution we specify for the bounce-back effect parameter λ is $N(0.30, 0.50^2)$. [9] The prior distributions for other parameters are the same as those in the Hamilton model without the bounce-back effect. In Table 3, the bounce-back effect is significant in the sense that the posterior mean of the bounce-back effect parameter is 0.10 and its 90% credible interval is $[0.03, 0.18]$. The posterior moments for other parameters are similar to those for the Hamilton model without the bounce-back effect (Model I) reported in Table 2.

Figure 5.A shows that incorporating the bounce-back effect in the Hamilton model results in a better correspondence between the posterior probabilities of recession and the NBER recession dates. It also provides a better correspondence between the posterior mean growth rates and the NBER regime-specific mean growth. However, the model completely misses

[9] Kim, Morley, and Piger (2005) found that the estimate of λ is 0.26 and its standard error is 0.06 via the maximum likelihood estimation method.

the 2001 recession, unlike the proposed model. [10] DIC decreases from 446.79 (Hamilton model) to 444.93 (extended Hamilton model). However, the decrease in DIC is only minor and it is still considerably higher than that for the proposed model (421.50).

Other than the 'bounce-back effect', we cannot preclude the possibility of an abrupt structural break in the long-run mean growth rate, as reported in the literature (e.g. Perron (1989) and Zivot and Andrews (2002)). We thus consider three additional models in which the random walk specification for the long-run mean growth rate in equation (30) is replaced by a structural shift in the long-run mean growth rate, which is modeled as a Markov-switching process with an absorbing state:

$$\delta_{D_t} = \delta_0(1 - D_t) + \delta_1 D_t, \tag{37}$$

where

$$Pr[D_t = 0|D_{t-1} = 0] = q_D, \quad Pr[D_t = 1|D_{t-1} = 1] = 1. \tag{38}$$

To summarize, we consider and estimate three additional models given below:

Model IV: Hamilton Model with a Structural Break in Long-Run Growth Rate

$$y_t = \delta_{D_t} + (1 - S_t)\mu_0 + S_t\mu_1 + e_t, \quad e_t \sim N(0, \sigma^2_{e,t})$$

$$\mu_0 > 0, \quad \mu_1 < 0$$

$$\pi_0\mu_0 + \pi_1\mu_1 = 0$$

Model V: Proposed Model with a Structural Break in Long-Run Growth Rate

$$y_t = \delta_{D_t} + (1 - S_t)\mu_{0,\tau} + S_t\mu_{1,\tau} + e_t, \quad e_t \sim N(0, \sigma^2_{e,t})$$

$$\mu_{0,\tau} > 0, \quad \mu_{1,\tau} < 0, \quad for\ all\ \tau$$

$$E[\pi_0\mu_{0,\tau} + \pi_1\mu_{1,\tau}] = 0$$

Model VI: Hamilton Model with a Bounce-Back Effect

[10] Figures 5.B-5.D depict posterior mean growth rates, stochastic volatility, and long-run mean growth rates over time, respectively. They are very similar to those for the Hamilton model without the bounce-back effect.

and a Structural Break in Long-Run Growth Rate

$$y_t = \delta_{D_t} + (1 - S_t)\mu_0 + S_t\mu_1 + \lambda \sum_{j=1}^{6} S_{t-j}(y_{t-j} - \delta_{D_{t-j}}) + e_t,$$

$$\mu_0 > 0, \quad \mu_1 < 0$$

$$\pi_0\mu_0 + \pi_1\mu_1 = 0.$$

The priors for the long-run growth rates before and after the break date are specified as $N(0.8, 0.5^2)$. The prior mean is based on the fact that the average growth rate for postwar U.S. real GDP is about 0.8 in the sample. The prior distribution for q_D is specified as $Beta(70, 1)$. The prior and the posterior moments for each of the above models, along with all related figures, are reported in the Appendix. A comparison of the DIC values reported in Table 4 gives the following result summary: i) The proposed model is most preferred; and ii) for each model, a random walk specification for the long-run mean growth rate is preferred to a one-time structural-break specification. [11] Our results confirm Stock and Watson's (2012) observation that the long-run mean growth rate of real output has declined steadily over the postwar sample.

6. Summary and Suggestion for Further Studies

As an economy and its institutions and policies evolve over time, so do the dynamics of the business cycle. Over time, we thus may need a more sophisticated empirical model that is capable of capturing the changes in the dynamics of the business cycle. The Great Moderation, i.e., the stabilization of the economy since the mid-1980s, is an example of such change. However, what is sometimes overlooked in empirical models of the business cycle is that the postwar booms or recessions are not all alike. For example, a two-state Markov-switching model of the business cycle, as proposed by Hamilton (1989), assumes that mean growth rates during all episodes of boom or recession are the same. While this assumption

[11] We also considered models in which we allow for a one-time structural break in the variance of the disturbance term e_t, as an alternative to the stochastic volatility. Even though we do not report the estimation results in the paper, we note that these models are inferior to those with stochastic volatility based on DIC.

may be valid for particular sample periods, it may not be realistic for a sample that covers the entire postwar period. This is why the original Hamilton model fails to provide sharp inferences on two distinctive business cycle regimes when the sample period is extended beyond that used by Hamilton (1989).

The extensions to the original Markov-switching approach of Hamilton (1989) include the introduction of a random walk hierarchical prior for each regime-specific mean growth rate and the inclusion of a cointegrating relationship between the average growth rates in recessionary and expansionary periods. By applying the propose approach to the postwar U.S. real GDP growth data from 1947:Q4 to 2011:Q3, we find three important features of the U.S. business cycle. First, the postwar booms and recessions are not all alike. Second, the long-run mean growth rate of real output has steadily and gradually declined over the postwar sample. Third, the error-correction mechanism works asymmetrically when the economy deviates from its long-run equilibrium.

The model presented in this paper may be further extended to the case of time-varying transition probabilities.

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Table 1. Prior and Posterior Moments: Hamilton Model

$$y_t = \delta_t + (1 - S_t)\mu_0 + S_t\mu_1 + e_t, \quad e_t \sim N(0, \sigma^2_{e,t})$$

$$\delta_t = \delta_{t-1} + \epsilon_t, \quad \epsilon_t \sim i.i.d.N(0, \sigma^2_\epsilon),$$

$$ln(\sigma^2_{e,t}) = ln(\sigma^2_{e,t-1}) + \eta_t, \quad \eta_t \sim i.i.d.N(0, \sigma^2_\eta),$$

$$Pr[S_t = 0|S_{t-1} = 0] = q, \quad Pr[S_t = 1|S_{t-1} = 1] = p$$

$$\pi_0\mu_0 + \pi_1\mu_1 = 0$$

where π_0 and π_1 are the unconditional probabilities.

	Prior		Posterior		
	Mean	SD	Mean	SD	90% Bands
μ_0	0.3000	0.5000	0.2262	0.0831	$[0.0958, 0.3680]$
σ^2_ϵ	0.0004	5.659e-006	0.00040	5.672e-006	$[0.00039, 0.00041]$
σ^2_η	0.3333	0.2352	0.0152	0.0071	$[0.0065, 0.0290]$
q	0.9000	0.0900	0.9526	0.0198	$[0.9172, 0.9812]$
p	0.8000	0.1212	0.7380	0.0837	$[0.5862, 0.8601]$
DIC	446.79				

Table 2. Prior and Posterior Moments: Proposed Model

$$y_t = \delta_t + (1 - S_t)\mu_{0,\tau} + S_t\mu_{1,\tau} + e_t, \quad e_t \sim N(0, \sigma^2_{e,t})$$

$$\mu_{0,\tau} = \mu_{0,\tau-1} + \theta_0(\pi_0\mu_{0,\tau-1} + \pi_1\mu_{1,\tau-1}) + \omega_{0,\tau}, \quad \omega_{0,\tau} \sim i.i.d.N(0, (1 - S_t)\sigma^2_{\omega,0})$$

$$\mu_{1,\tau} = \mu_{1,\tau-1} + \theta_1(\pi_0\mu_{0,\tau-1} + \pi_1\mu_{1,\tau-1}) + \omega_{1,\tau}, \quad \omega_{1,\tau} \sim i.i.d.N(0, S_t\sigma^2_{\omega,1})$$

$$\delta_t = \delta_{t-1} + \epsilon_t, \quad \epsilon_t \sim i.i.d.N(0, \sigma^2_\epsilon),$$

$$ln(\sigma^2_{e,t}) = ln(\sigma^2_{e,t-1}) + \eta_t, \quad \eta_t \sim i.i.d.N(0, \sigma^2_\eta),$$

$$Pr[S_t = 0|S_{t-1} = 0] = q, \quad Pr[S_t = 1|S_{t-1} = 1] = p$$

$$\mu_{0,\tau} > 0, \quad \mu_{1,\tau} < 0, \quad for\ all\ \tau,$$

$$-1 < 1 + \theta_0\pi_0 + \theta_1\pi_1 < 1,$$

where π_0 and π_1 are the unconditional probabilities.

	Prior		Posterior		
	Mean	SD	Mean	SD	90% Bands
θ_0	-0.1000	0.5000	-0.2924	0.2658	$[-0.7826, 0.0769]$
θ_1	-0.1000	0.5000	-0.1163	0.3842	$[-0.7901, 0.4675]$
$\sigma^2_{\omega 0}$	0.2500	0.1450	0.1195	0.0445	$[0.0663, 0.2029]$
$\sigma^2_{\omega 1}$	0.2500	0.1450	0.1476	0.0585	$[0.0777, 0.2590]$
σ^2_ϵ	0.0004	5.659e-006	0.0004	5.673e-006	$[0.00039, 0.00041]$
σ^2_η	0.3333	0.2352	0.0141	0.0072	$[0.0061, 0.0287]$
q	0.9000	0.0900	0.9204	0.0267	$[0.8723, 0.9577]$
p	0.8000	0.1212	0.7848	0.0595	$[0.6789, 0.8739]$
DIC	421.50				

Table 3. Prior and Posterior Moments: Hamilton Model with 'Bounce-bak Effect'

$$y_t = \delta_t + (1 - S_t)\mu_0 + S_t\mu_1 + \lambda \sum_{j=1}^{6} S_{t-j}(y_{t-j} - \delta_{t-j}) + e_t, \quad e_t \sim N(0, \sigma^2_{e,t})$$

$$\delta_t = \delta_{t-1} + \epsilon_t, \quad \epsilon_t \sim i.i.d.N(0, \sigma^2_\epsilon),$$

$$ln(\sigma^2_{e,t}) = ln(\sigma^2_{e,t-1}) + \eta_t, \quad \eta_t \sim i.i.d.N(0, \sigma^2_\eta),$$

$$Pr[S_t = 0|S_{t-1} = 0] = q, \quad Pr[S_t = 1|S_{t-1} = 1] = p$$

$$\pi_0\mu_0 + \pi_1\mu_1 = 0$$

where π_0 and π_1 are the unconditional probabilities.

	Prior		Posterior		
	Mean	SD	Mean	SD	90% Bands
μ_0	0.3000	0.5000	0.2950	0.0885	$[0.1532, 0.4450]$
σ^2_ϵ	0.0004	5.659e-006	0.0004	5.669e-006	$[0.00039, 0.00041]$
σ^2_η	0.3333	0.2352	0.0129	0.0058	$[0.0062, 0.0240]$
q	0.9000	0.0900	0.9466	0.0183	$[0.9136, 0.9729]$
p	0.8000	0.1212	0.7666	0.0688	$[0.6461, 0.8703]$
λ	0.3000	0.5000	0.1041	0.0454	$[0.0346, 0.1816]$
DIC	444.93				

Table 4. Deviance Information Criterion (DIC): Model Comparison

Model	DIC Value
Model I	446.79
Model II	**421.50**
Model III	444.93
Model IV	**441.99**
Model V	453.59
Model VI	452.79

Note 1

Model I: Hamilton model;

Model II: Proposed model;

Model III: Hamilton model with a bounce-back effect;

Model IV: Proposed model with a structural break in the long-run growth rate;

Model V: Hamilton model with a structural break in the long-run growth rate;

Model VI: Hamilton model with a bounce-back effect and a structural break in the long-run growth rate.

Note 2

The DIC value is calculated using

$$DIC = -2E_{\Psi|y}[\log f(y|\Psi)] + 2\left\{\log f(y|\overline{\Psi}) - E_{\Psi|y}\left[\log f(y|\Psi)\right]\right\}$$

and DIC prefers the model with a small value.

Figure 1. Real GDP Growth and Its Episode-Specific Means During NBER Booms and Recessions [1947:IV - 2011:III]

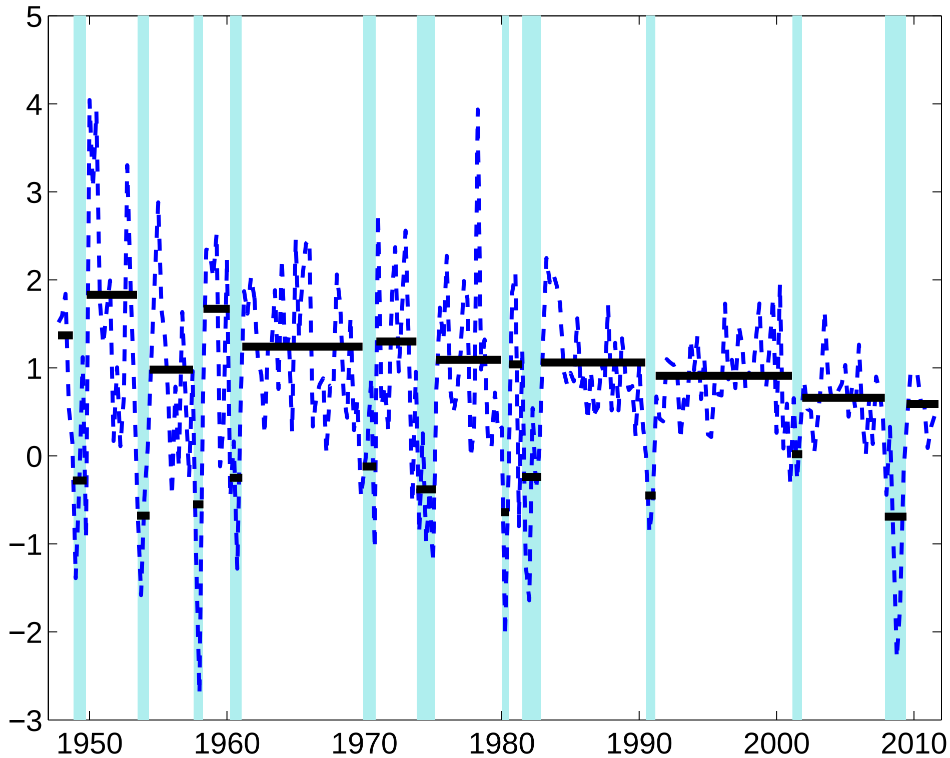
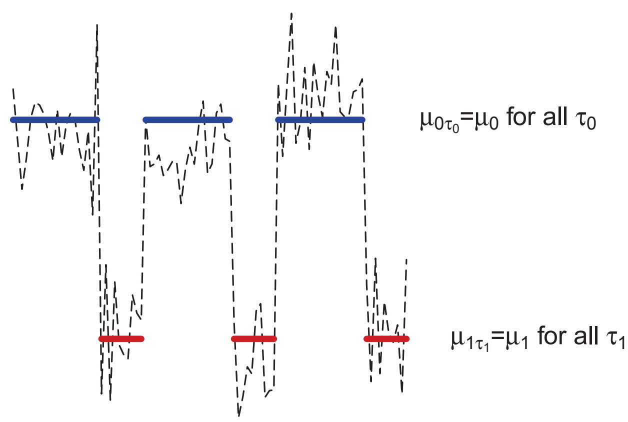
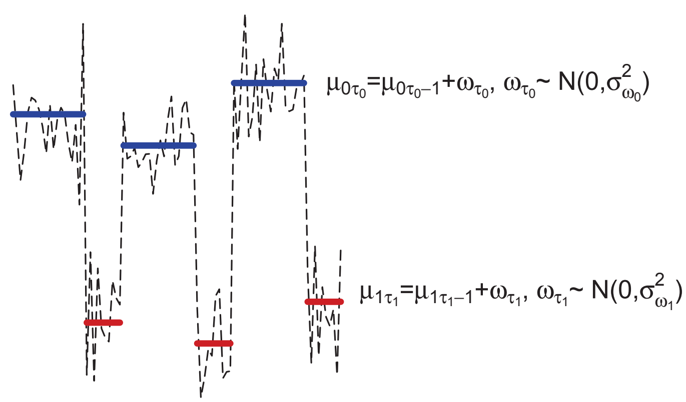


Figure 2. Comparison of Hamilton (1989) Model and the Proposed Model

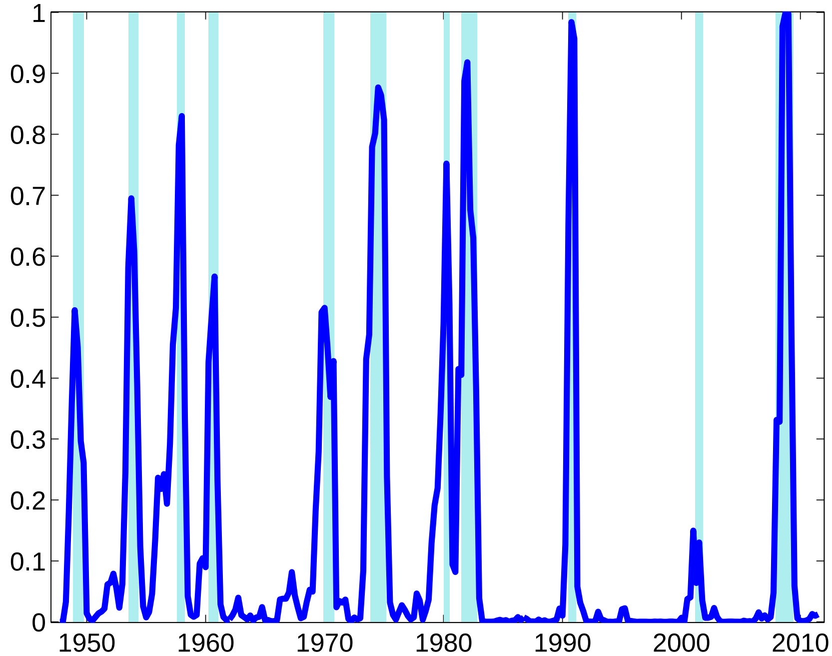


A. Hamilton Model

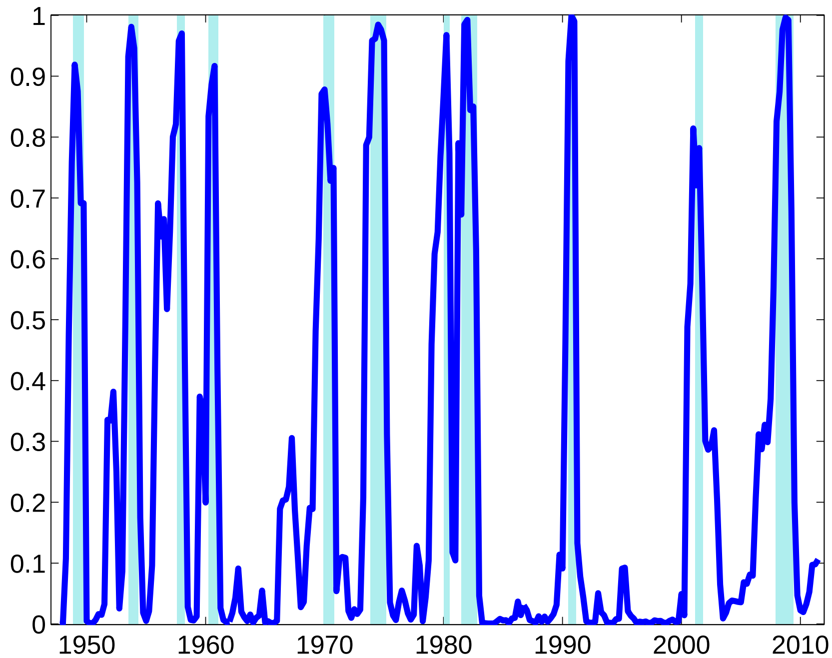


B. Proposed Model

Figure 3. Posterior Probabilities of Recession: Hamilton Model vs. Proposed Model

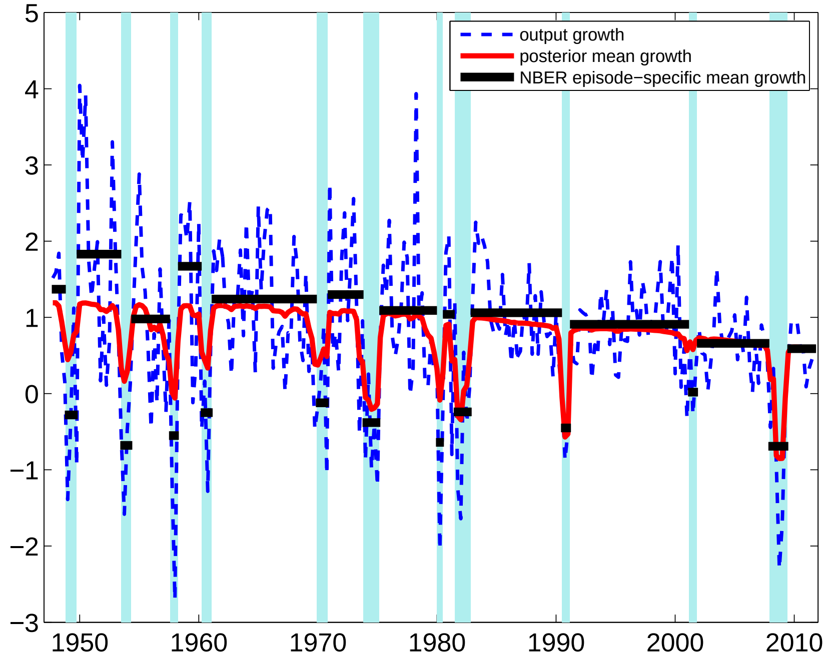


A. Model I: Hamilton Model

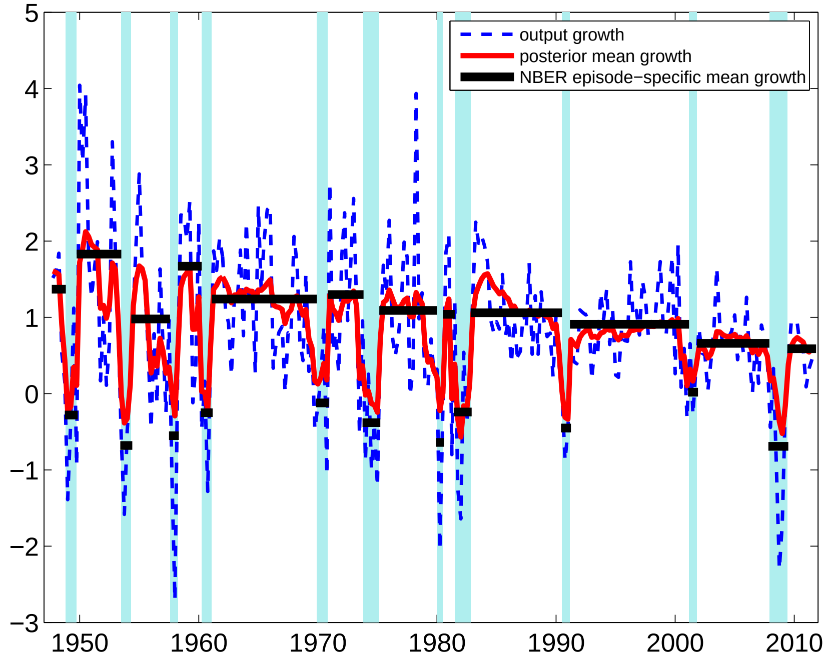


B. Model II: Proposed Model

Figure 4. Posterior Mean Growth Rates: Hamilton Model vs. Proposed Model

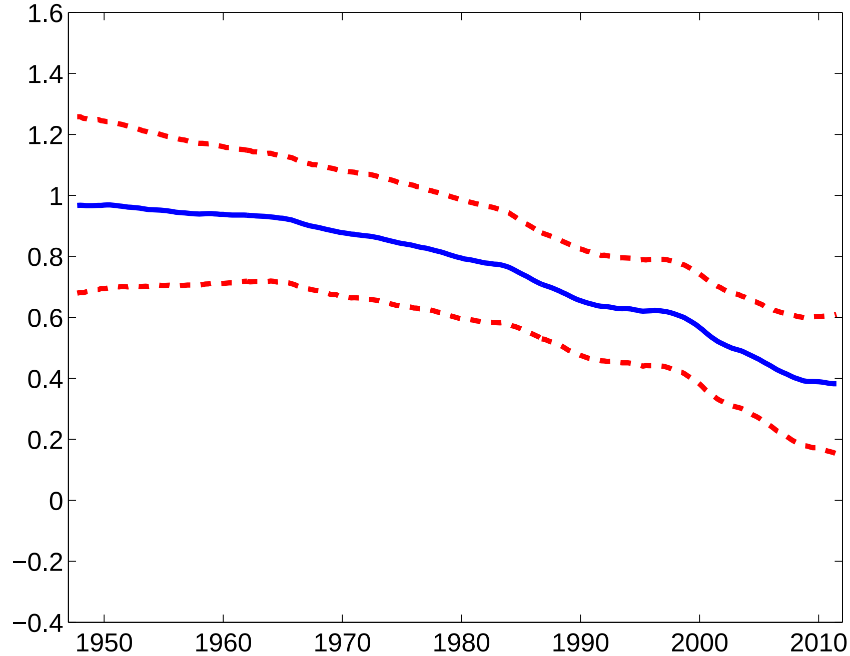


A. Model I: Hamilton Model

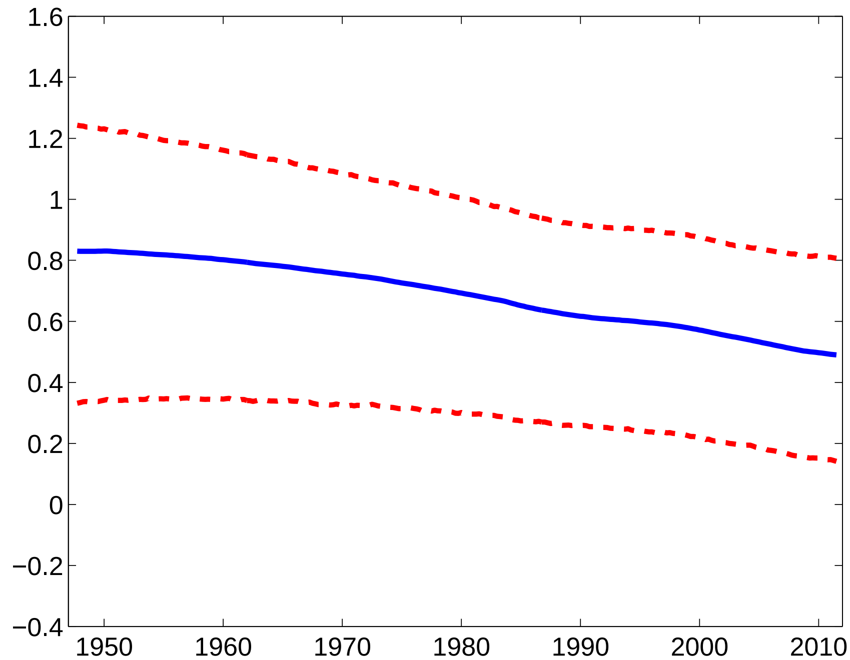


B. Model II: Proposed Model

Figure 5. Posterior Long-run Mean Growth Rates with 90% Posterior Bands: Hamilton Model vs. Proposed Model

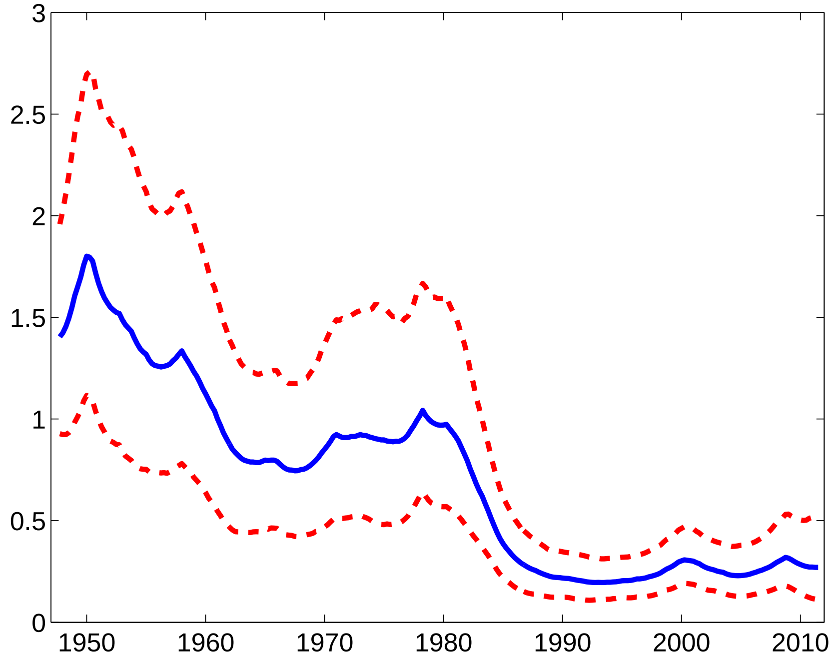


A. Model I: Hamilton Model

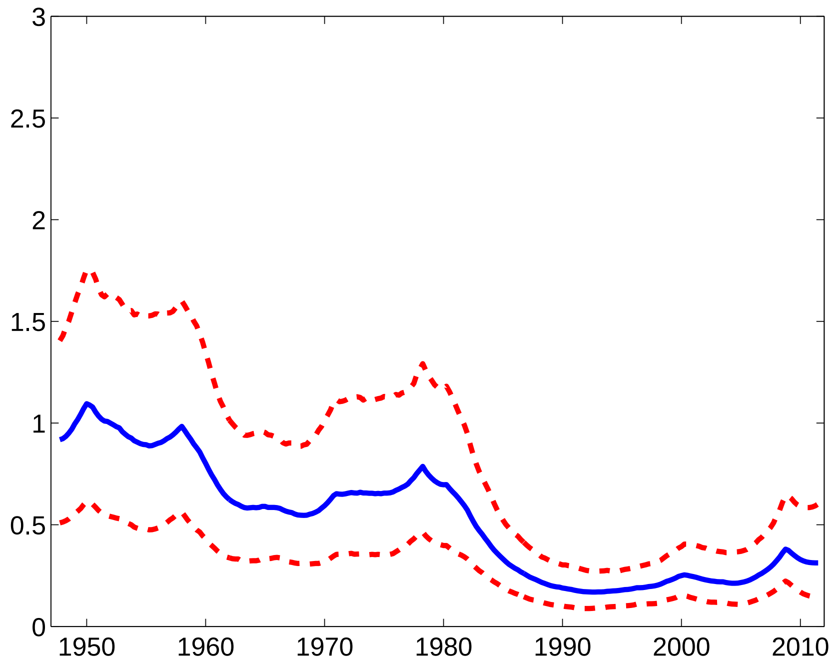


B. Model II: Proposed Model

Figure 6. Posterior Volatilities with 90% Posterior Bands: Hamilton Model vs. Proposed Model

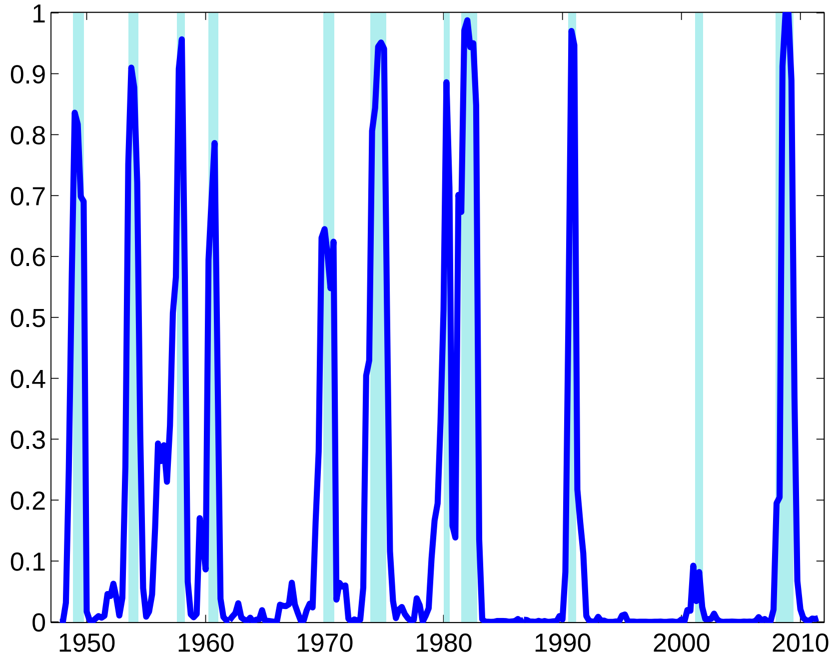


A. Model I: Hamilton Model

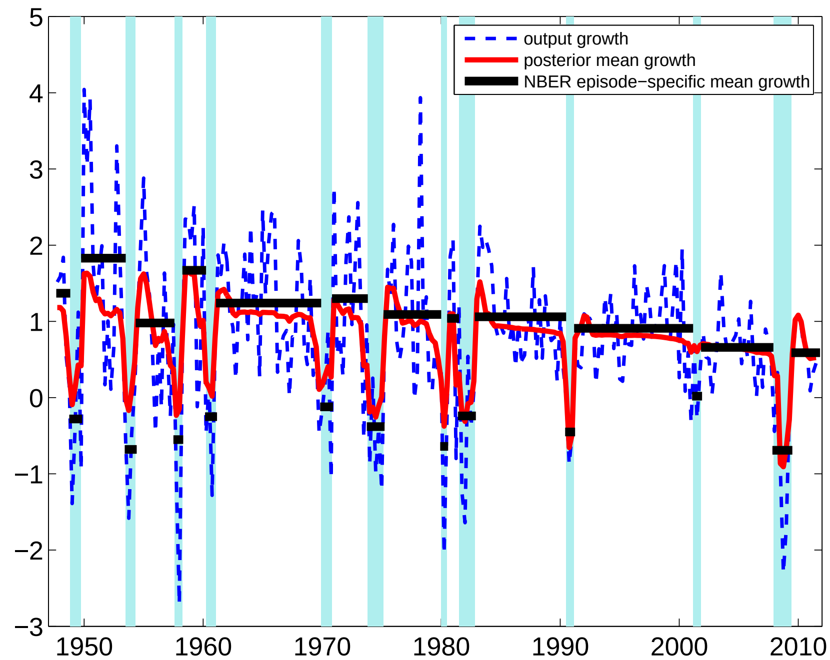


B. Model II: Proposed Model

Figure 7. Posterior Probabilities of Recession and Posterior Mean Growth Rates: Hamilton Model with a 'Bounce-back Effect'



A. Posterior Probabilities of Recession



B. Mean Growth Rates

Supplement to "Markov-Switching Models with Evolving Regime-Specific Parameters: Are Postwar Booms or Recessions All Alike?"

Yunjong Eo [1]
University of Sydney
and
Chang-Jin Kim [2]
University of Washington and Korea University

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[1] **Eo:** School of Economics, University of Sydney, Sydney, NSW 2006, Australia [e-mail: yunjong.eo@sydney.edu.au]

[2] **Kim:** (Corresponding Author) Department of Economics, University of Washington, Seattle, WA. 98195, U.S.A. [e-mail: changjin@u.washington.edu] and Department of Economics, Korea University, Seoul 136-701, Korea

Appendix A. Details of the MCMC Procedure

A.1. Generating $\tilde{\mu}^*_{0,T}$ and $\tilde{\mu}^*_{1,T}$ conditional on $\tilde{S}_T$, parameters $\tilde{\psi}$, and data $\tilde{Y}_T$.

Conditional on $\tilde{S}_T$, equations (23)-(24) form a linear state-space model for the extended model in Section 3.1. This allows us to employ a slightly modified version of the procedure proposed by Carter and Kohn (1994). The conditional joint posterior distribution of $\tilde{\mu}^*_{0,T}$ and $\tilde{\mu}^*_{1,T}$ can be decomposed as:

$$p(\tilde{\mu}^*_{0,T}, \tilde{\mu}^*_{1,T} | \tilde{Y}_T, \tilde{S}_T, \tilde{\Psi}) = p(\mu^*_{0,T}, \mu^*_{1,T} | \tilde{Y}_T, \tilde{S}_T, \tilde{\psi}) \prod_{t=1}^{T-1} p(\mu^*_{0,t}, \mu^*_{1,t} | \mu^*_{0,t+1}, \mu^*_{1,t+1}, \tilde{Y}_t, \tilde{S}_T, \tilde{\psi}), \quad (A.1)$$

which suggests that we can sequentially generate $\mu^*_{0,t}$ and $\mu^*_{1,t}$ for $t = T, T-1, \ldots, 2, 1$. Note that, for identification of the model, we need to impose the restrictions, $\mu^*_{0,t} > 0$ and $\mu^*_{1,t} < 0$ for all t.

We run the Kalman filter for the state-space model given by equations (23)-(24) in order to obtain and save $\mu^*_{t|t} = E(\mu^*_t | \tilde{Y}_t, \tilde{S}_t, \tilde{\psi})$ and $P_{t|t} = Cov(\mu^*_t | \tilde{Y}_t, \tilde{S}_t, \tilde{\psi})$ for $t = 1, 2, ..., T$, where $\tilde{Y}_t = [\, y_1 \quad y_2 \quad \ldots \quad y_t \,]'$.

For $t = T$, we generate $\mu^*_T = [\, \mu^*_{0,T} \quad \mu^*_{1,T} \,]'$ from the joint normal distribution

$$\mu^*_T \; | \tilde{Y}_T, \tilde{S}_T, \tilde{\psi} \;\sim\; N(\mu^*_{T|T}, P_{T|T}). \quad (A.2)$$

For $t = T-1, T-2, .., 1$, we generate $\mu^*_t = [\, \mu^*_{0,t} \quad \mu^*_{1,t} \,]'$ conditional on $\mu^*_{t+1} = [\, \mu^*_{0,t+1} \quad \mu^*_{1,t+1} \,]'$. For this purpose, we first calculate

$$\mu^*_{t|t,\mu^*_{t+1}} = E(\mu^*_t | \tilde{Y}_t, \mu^*_{t+1}, \tilde{S}_T, \tilde{\psi}) = \mu^*_{t|t} + P_{t|t} F'_{t+1} (F_{t+1} P_{t|t} F'_{t+1} + \Omega_{t+1})^{-1} (\mu^*_{t+1} - F_{t+1} \mu^*_{t|t}) \quad (A.3)$$

and

$$P_{t|t,\mu_{t+1}} = Cov(\mu^*_t | \tilde{Y}_t, \mu^*_{t+1}, \tilde{S}_T, \tilde{\psi}) = P_{t|t} - P_{t|t} F'_{t+1} (F_{t+1} P_{t|t} F'_{t+1} + \Omega_{t+1})^{-1} F_{t+1} P_{t|t}. \quad (A.4)$$

Then, we can generate $\mu^*_{0,t}$ and $\mu^*_{1,t}$ in the following way:

i) If $S_t = 0$ and $S_{t+1} = 1$, we set $\mu^*_{0,t} =$ (1,1) element of $\mu^*_{t|t,\mu^*_{t+1}}$, and generate $\mu^*_{1,t}$ from the following distribution:

$$\mu^*_{1,t}|\mu^*_{t+1}, \tilde{Y}_t, \tilde{S}_T, \tilde{\psi} \sim N(\mu^*_{t|t,\mu^*_{t+1}}(2,1),\ P_{t|t,\mu^*_{t+1}}(2,2)), \qquad (A.5)$$

where $\mu^*_{t|t,\mu^*_{t+1}}(2,1)$ and $P_{t|t,\mu^*_{t+1}}(2,2)$ are the (2,1) element of $\mu^*_{t|t,\mu^*_{t+1}}$ and the (2,2) element of $P_{t|t,\mu^*_{t+1}}$, respectively.

ii) If $S_t = 1$ and $S_{t+1} = 0$, we set $\mu^*_{1,t}$ = (2,1) element of $\mu^*_{t|t,\mu^*_{t+1}}$, and generate $\mu^*_{0,t}$ from the following distribution:

$$\mu^*_{0,t}|\mu^*_{t+1}, \tilde{Y}_t, \tilde{S}_T, \tilde{\psi} \sim N(\mu^*_{t|t,\mu^*_{t+1}}(1,1),\ P_{t|t,\mu^*_{t+1}}(1,1)), \qquad (A.6)$$

where $\mu^*_{t|t,\mu^*_{t+1}}(1,1)$ and $P_{t|t,\mu^*_{t+1}}(1,1)$ are the (1,1) element of $\mu^*_{t|t,\mu^*_{t+1}}$ and the (1,1) element of $P_{t|t,\mu^*_{t+1}}$, respectively.

iii) Otherwise, we set $\mu^*_{0,t}$ = (1,1) element of $\mu^*_{t|t,\mu^*_{t+1}}$ and $\mu^*_{1,t}$ = (2,1) element of $\mu^*_{t|t,\mu^*_{t+1}}$.

A.2. Generating $\tilde{S}_T$ conditional on $\tilde{\mu}^*_{0,T}$, $\tilde{\mu}^*_{1,T}$, parameters $\tilde{\psi}$, and data $\tilde{Y}_T$

We employ a modified version of Albert and Chib's (1993) single-move Gibbs sampling for generating S_t, $t = 1, 2, ..., T$, conditional on $\tilde{S}_{\neq t} = [\,S_1 \quad \ldots S_{t-1} \quad S_{t+1} \quad \ldots \quad S_T\,]'$ and other variates. The key is in appropriately evaluating the predictive densities of y_t under two possible alternative regimes at time t (i.e., for $S_t = 0$ and for $S_t = 1$). However, unlike in the Hamilton (1989) model with constant mean growth rates (μ_0 and μ_1), the mean growth rates during recessions or booms in our model are not always defined, as discussed in the earlier sections. For example, conditional on $S_t = 1$ in the $(j-1)-th$ iteration of the MCMC procedure, only μ_{1,τ_1} is defined and μ_{0,τ_1} is not. The difficulty is that, when evaluating the predictive densities of y_t under two alternative regimes at the $j-th$ iteration of the MCMC procedure, we need μ_{0,τ_1} as well as μ_{1,τ_1}. We overcome this difficulty by taking advantage of hierarchical priors in (20)-(21). Note that $\mu^*_{0,t}$ and $\mu^*_{1,t}$ in equations (20)-(21) summarize the hierarchical priors for the mean growth rates under two alternative regimes, for all t.

Thus, the method for generating $\tilde{S}_t$ conditional on $\tilde{S}_{\neq t}$ and other variates is the same as in Albert and Chib (1993), except that we use $\mu^*_{0,t}$ and $\mu^*_{1,t}$ as the mean growth rates under two possible alternative regimes at each point in time. As in Albert and Chib (1993), $p(S_t|\tilde{Y}_T, \tilde{S}_{\neq t}, \tilde{\mu}^*_{0,T}, \tilde{\mu}^*_{1,T}, \tilde{\psi})$ can be derived as:

$$p(S_t|\tilde{Y}_T, \tilde{S}_{\neq t}, \tilde{\mu}^*_{0,T}, \tilde{\mu}^*_{1,T}, \tilde{\psi}) \propto Pr(S_t|S_{t-1})Pr(S_{t+1}|S_t)p(y_t|\tilde{Y}_{t-1}, S_t, \mu^*_{0,t}, \mu^*_{1,t}, \tilde{\psi}), \qquad (A.7)$$

where

$$p(y_t|\tilde{Y}_{t-1}, S_t, \mu^*_{0,t}, \mu^*_{1,t}, \tilde{\psi}) = \frac{1}{\sqrt{2\pi\sigma_e^2}} exp\left(-\frac{1}{2\sigma_e^2}(y_t - \delta - \mu^*_{S_t,t})^2\right). \qquad (A.8)$$

Then, S_t can be generated from

$$Pr[S_t = 1|\tilde{Y}_T, \tilde{S}_{\neq t}, \tilde{\mu}^*_{0,T}, \tilde{\mu}^*_{1,T}, \tilde{\psi})] = \frac{p(S_t = 1|\tilde{Y}_T, \tilde{S}_{\neq t}, \tilde{\mu}^*_{0,T}, \tilde{\mu}^*_{1,T}, \tilde{\psi})}{\sum_{j=0}^{1} p(S_t = j|\tilde{Y}_T, \tilde{S}_{\neq t}, \tilde{\mu}^*_{0,T}, \tilde{\mu}^*_{1,T}, \tilde{\psi})}. \qquad (A.9)$$

Note that, in Albert and Chib's (1993) procedure for the Hamilton model, they have $\mu^*_{S_t,t} = \mu_{S_t}$, $S_t = 0, 1$.

A.3. Generating θ_0, θ_1, $\sigma^2_{\omega,0}$ and $\sigma^2_{\omega,1}$, conditional on $\tilde{\mu}^*_{0,T}$, $\tilde{\mu}^*_{1,T}$, and $\tilde{S}_T$

For given $\tilde{S}_T$, we first extract $\tilde{\mu}_{0,N} = [\, \mu_{0,1} \;\; \ldots \;\; \mu_{0,N} \,]'$ and $\tilde{\mu}_{1,N} = [\, \mu_{1,1} \;\; \ldots \;\; \mu_{1,N} \,]'$, from $\tilde{\mu}^*_{0,T} = [\, \mu^*_{0,1} \;\; \ldots \;\; \mu^*_{0,T} \,]'$ and $\tilde{\mu}^*_{1,T} = [\, \mu^*_{1,1} \;\; \ldots \;\; \mu^*_{1,T} \,]'$, as implied by the equivalence of equations (4)-(5) and (7)-(8). For example, $\tilde{\mu}_{0,N}$ and $\tilde{\mu}_{1,N}$ are the collections of $\mu^*_{0,t}$'s and $\mu^*_{1,t}$'s whenever $S_t \neq S_{t-1}$.

Then, based on equations (14)-(15), θ_0 and θ_1 can be generated conditional on $\sigma^2_{\omega,0}$ and $\sigma^2_{\omega,1}$; and then $\sigma^2_{\omega,0}$ and $\sigma^2_{\omega,1}$ can be generated conditional on θ_0 and θ_1. The prior and posterior distributions for generating these parameters are described below.

<u>*Prior*</u>

$$\theta_j \sim N(\underline{\theta}_j, \underline{\Sigma}_{\theta_j}), \quad j = 0, 1 \qquad (A.10)$$

$$\sigma^2_{\omega,j} \;\sim\; IG\left(\frac{\nu_{\omega,j}}{2}, \frac{h_{\omega,j}}{2}\right), \quad j = 0, 1, \qquad (A.11)$$

<u>*Posterior*</u>

$$\theta_j \mid \tilde{\mu}^*_{0,T}, \tilde{\mu}^*_{1,T}, \tilde{S}_T, \sigma^2_{\omega,0}, \sigma^2_{\omega,1} \sim N(\bar{\theta}_j, \bar{\Sigma}_{\theta_j}), \quad j = 0, 1, \qquad (A.12)$$

$$\sigma^2_{\omega,j} \mid \theta_j, \tilde{\mu}^*_{0,T}, \tilde{\mu}^*_{1,T}, \tilde{S}_T \ \sim IG\left(\frac{\nu_{\omega,j}+N_j}{2}, \frac{h_{\omega,j}+\sum_{\tau\in\Gamma_j}\omega^2_{j,\tau_j}}{2}\right), \quad j=0,1, \qquad (A.13)$$

where

$$\bar{\theta}_j = \bar{\Sigma}_{\theta_j}\left(\underline{\Sigma}^{-1}_{\theta_j}\underline{\theta}_j + \frac{1}{\sigma^2_{\omega,j}}\sum_{\tau\in\Gamma_j}(\pi_i\mu_{i,\tau-1}+\pi_j\mu_{j,\tau-1})(\mu_{j,\tau}-\mu_{j,\tau-1})\right), \qquad (A.14)$$

$$(j,i)=(0,1),\ (1,0)$$

$$\bar{\Sigma}_{\theta_j} = \left(\underline{\Sigma}^{-1}_{\theta_j} + \frac{1}{\sigma^2_{\omega,j}}\sum_{\tau\in\Gamma_j}(\pi_i\mu_{i,\tau-1}+\pi_j\mu_{j,\tau-1})^2\right)^{-1}, \qquad (A.15)$$

$$\omega_{j,\tau} = \mu_{j,\tau}-\mu_{j,\tau-1}-\theta_j(\pi_i\mu_{i,\tau-1}+\pi_j\mu_{j,\tau-1}), \quad (j,i)=(0,1),(1,0), \qquad (A.16)$$

and Γ_j is a collection of the regime episode τ's which belong to the regime j where $j=0$ for boom episodes and $j=1$ for recession episodes.

A.4. Generating δ and σ^2_e, conditional on $\tilde{\mu}^*_{0,T}$, $\tilde{\mu}^*_{1,T}$, $\tilde{S}_T$, and $\tilde{Y}_T$

This step is based on equation (19). Conditional on $\tilde{S}_T$, $\tilde{\mu}^*_{0,T}$, $\tilde{\mu}^*_{1,T}$ and $\tilde{Y}_T$, we define $y^*_t = y_t-(1-S_t)\mu^*_{0,t}-S_t\mu^*_{1,t}$, $t=1,2,...,T$. Then, we have $y^*_t=\delta+e_t$. Based on this, the conditional posterior distributions for the δ and σ^2_e parameters can be easily derived. The prior and posterior distributions are given below:

<u>*Prior*</u>

$$\delta \sim N(\underline{\delta}, \underline{\Sigma}_\delta), \qquad (A.17)$$

$$\sigma^2_e \ \sim\ IG\left(\frac{\nu_e}{2}, \frac{h_e}{2}\right), \quad j=0,1, \qquad (A.18)$$

<u>*Posterior*</u>

$$\delta \mid \tilde{\mu}^*_{0,T}, \tilde{\mu}^*_{1,T}, \tilde{S}_T, \sigma^2_e, \tilde{y}_T \ \sim N(\bar{\delta}, \bar{\Sigma}_\delta), \qquad (A.19)$$

$$\sigma_e^2 \mid \delta, \tilde{\mu}^*_{0,T}, \tilde{\mu}^*_{1,T}, \tilde{S}_T, \tilde{Y}_T \sim IG\left(\frac{\nu_e + T}{2}, \frac{h_e + \sum_{t=1}^T (y_t^* - \delta)^2}{2}\right), \quad j = 0, 1, \tag{A.20}$$

where

$$\bar{\Sigma}_\delta = \left(\underline{\Sigma_\delta}^{-1} + \frac{T}{\sigma_e^2}\right)^{-1} \tag{A.21}$$

and

$$\bar{\delta} = \bar{\Sigma}_\delta \left(\underline{\Sigma_\delta}^{-1} \underline{\delta} + \frac{1}{\sigma_e^2} \sum_{t=1}^T y_t^*\right). \tag{A.22}$$

A.5. Generating q and p conditional on $\tilde{S}_T$

We employ the following Beta priors for q and p:

Prior

$$q \sim Beta(u_{00}, u_{01}), \tag{A.23}$$

$$p \sim Beta(u_{11}, u_{10}), \tag{A.24}$$

where u_{ij}, $i, j = 0, 1$, are the hyper-parameters. Then the posterior distribution can be derived as:

Posterior

$$p|\tilde{S}_T \sim Beta(u_{11} + n_{11}, u_{10} + n_{10}), \tag{A.25}$$

$$q|\tilde{S}_T \sim Beta(u_{00} + n_{00}, u_{01} + n_{01}), \tag{A.26}$$

where n_{ij} refers to the total number of transitions from state i to state j.

Appendix B. Tables

Table B.1. Episode-Specific Mean Growth Rates of Real GDP During NBER Booms and Recessions [1947:Q4 - 2011:Q3]

Boom		Recession	
47:Q4 ∼ 48:Q3	1.37	48:Q4 ∼ 49:Q4	-0.28
50:Q1 ∼ 53:Q2	1.83	53:Q3 ∼ 54:Q2	-0.64
54:Q3 ∼ 57:Q2	0.98	57:Q3 ∼ 58:Q2	-0.55
58:Q3 ∼ 60:Q1	1.67	60:Q2 ∼ 61:Q1	-0.25
61:Q2 ∼ 69:Q3	1.24	69:Q4 ∼ 70:Q4	-0.12
71:Q1 ∼ 73:Q3	1.30	73:Q4 ∼ 75:Q1	-0.38
75:Q2 ∼ 79:Q4	1.09	80:Q1 ∼ 80:Q3	-0.64
80:Q4 ∼ 81:Q2	1.04	81:Q3 ∼ 82:Q4	-0.24
83:Q1 ∼ 90:Q2	1.06	90:Q3 ∼ 91:Q1	-0.45
91:Q2 ∼ 00:Q4	0.91	01:Q1 ∼ 01:Q4	0.02
02:Q1 ∼ 07:Q3	0.66	07:Q4 ∼ 09:Q2	-0.69
09:Q3 ∼ 11:Q3	0.59		
Mean	1.15		-0.38
Maximum	1.83		0.02
Minimum	0.59		-0.69
Standard Deviation	0.37		0.23

Table B.2. Model IV: Hamilton Model with a Structural Break in δ

$$y_t = \delta_{D_t} + (1 - S_t)\mu_0 + S_t\mu_1 + e_t, \quad e_t \sim N(0, \sigma^2_{e,t})$$

$$ln(\sigma^2_{e,t}) = ln(\sigma^2_{e,t-1}) + \eta_t, \quad \eta_t \sim i.i.d.N(0, \sigma^2_\eta),$$

$$Pr[S_t = 0|S_{t-1} = 0] = q, \quad Pr[S_t = 1|S_{t-1} = 1] = p$$

$$Pr[D_t = 0|D_{t-1} = 0] = q_D, \quad Pr[D_t = 1|D_{t-1} = 1] = 1$$

$$\pi_0\mu_0 + \pi_1\mu_1 = 0$$

where π_0 and π_1 are the unconditional probabilities.

	Prior		Posterior		
	Mean	SD	Mean	SD	90% Bands
μ_0	0.3000	0.5000	0.2326	0.0953	[0.0907, 0.4018]
σ^2_η	0.3333	0.2352	0.0143	0.0070	[0.0066, 0.0274]
q	0.9000	0.0900	0.9520	0.0217	[0.9122, 0.9822]
p	0.8000	0.1212	0.7436	0.0840	[0.5920, 0.8644]
δ_0	0.8000	0.5000	0.9145	0.1515	[0.6956, 1.1852]
δ_1	0.8000	0.5000	0.4400	0.1377	[0.2113, 0.6651]
q_D	0.9859	0.0139	0.9913	0.0065	[0.9788, 0.9985]
DIC	453.59				

Table B.3. Model V: Proposed Model with a Structural Break in δ

$$y_t = \delta_{D_t} + (1 - S_t)\mu_{0,\tau} + S_t\mu_{1,\tau} + e_t, \quad e_t \sim N(0, \sigma^2_{e,t})$$

$$\mu_{0,\tau} = \mu_{0,\tau-1} + \theta_0(\pi_0\mu_{0,\tau-1} + \pi_1\mu_{1,\tau-1}) + \omega_{0,\tau}, \quad \omega_{0,\tau} \sim i.i.d.N(0, \sigma^2_{0,\tau})$$

$$\mu_{1,\tau} = \mu_{1,\tau-1} + \theta_1(\pi_0\mu_{0,\tau-1} + \pi_1\mu_{1,\tau-1}) + \omega_{1,\tau}, \quad \omega_{1,\tau} \sim i.i.d.N(0, \sigma^2_{1,\tau})$$

$$ln(\sigma^2_{e,t}) = ln(\sigma^2_{e,t-1}) + \eta_t, \quad \eta_t \sim i.i.d.N(0, \sigma^2_\eta),$$

$$Pr[S_t = 0|S_{t-1} = 0] = q, \quad Pr[S_t = 1|S_{t-1} = 1] = p$$

$$Pr[D_t = 0|D_{t-1} = 0] = q_D, \quad Pr[D_t = 1|D_{t-1} = 1] = 1$$

$$\mu_{0,\tau} > 0, \quad \mu_{1,\tau} < 0, \quad for\ all\ \tau,$$

$$-1 < 1 + \theta_0\pi_0 + \theta_1\pi_1 < 1,$$

where π_0 and π_1 are the unconditional probabilities.

	Prior		Posterior		
	Mean	SD	Mean	SD	90% Bands
θ_0	-0.1000	0.5000	-0.4322	0.3214	$[-1.0063, 0.0270]$
θ_1	-0.1000	0.5000	-0.2533	0.4549	$[-1.0069, 0.4901]$
$\sigma^2_{\omega 0}$	0.2500	0.1450	0.1259	0.0467	$[0.0700, 0.2119]$
$\sigma^2_{\omega 1}$	0.2500	0.1450	0.1786	0.0749	$[0.0923, 0.3189]$
σ^2_η	0.3333	0.2352	0.0142	0.0069	$[0.0073, 0.0264]$
q	0.9000	0.0900	0.9382	0.0253	$[0.8946, 0.9731]$
p	0.8000	0.1212	0.7605	0.0753	$[0.6235, 0.8676]$
δ_0	0.8000	0.5000	1.0031	0.1594	$[0.7243, 1.2485]$
δ_1	0.8000	0.5000	0.3542	0.1384	$[0.1243, 0.5880]$
q_D	0.9859	0.0139	0.9910	0.0075	$[0.9772, 0.9982]$
DIC	441.99				

Table B.4. Model VI: Hamilton Model with a Bounce-back Effect and a Structural Break in δ

$$y_t = \delta_{D_t} + (1 - S_t)\mu_0 + S_t\mu_1 + \lambda \sum_{j=1}^{6} S_{t-j}(y_{t-j} - \delta_{t-j}) + e_t, \quad e_t \sim N(0, \sigma^2_{e,t})$$

$$ln(\sigma^2_{e,t}) = ln(\sigma^2_{e,t-1}) + \eta_t, \quad \eta_t \sim i.i.d.N(0, \sigma^2_\eta),$$

$$Pr[S_t = 0|S_{t-1} = 0] = q, \quad Pr[S_t = 1|S_{t-1} = 1] = p$$

$$Pr[D_t = 0|D_{t-1} = 0] = q_D, \quad Pr[D_t = 1|D_{t-1} = 1] = 1$$

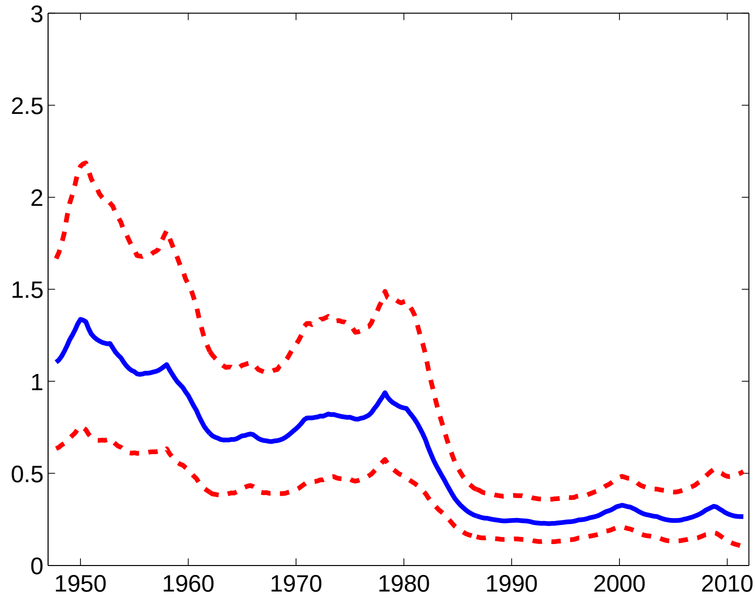
$$\pi_0\mu_0 + \pi_1\mu_1 = 0$$

where π_0 and π_1 are the unconditional probabilities.

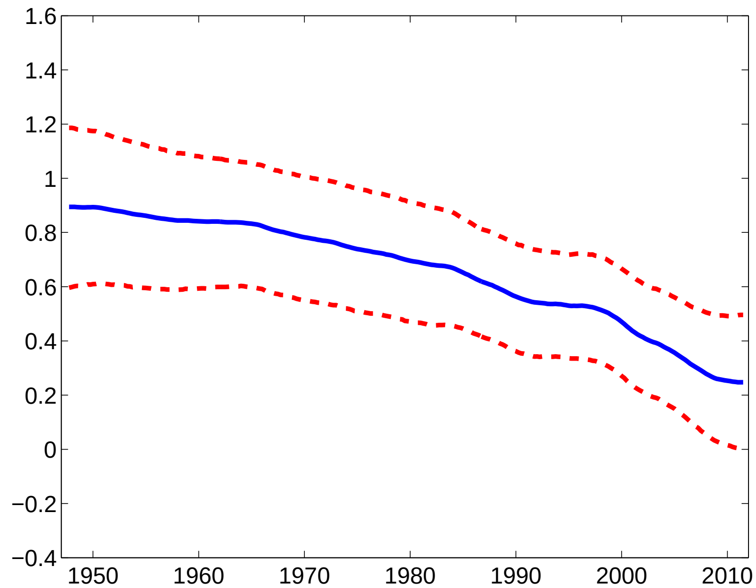
	Prior		Posterior		
	Mean	SD	Mean	SD	90% Bands
μ_0	0.3000	0.5000	0.2697	0.0891	$[0.1269, 0.4194]$
σ^2_η	0.3333	0.2352	0.0135	0.0067	$[0.0062, 0.0269]$
q	0.9000	0.0900	0.9492	0.0195	$[0.9143, 0.9767]$
p	0.8000	0.1212	0.7703	0.0718	$[0.6423, 0.8755]$
λ	0.3000	0.5000	0.0951	0.0523	$[0.0123, 0.1843]$
δ_0	0.8000	0.5000	0.7761	0.1403	$[0.5670, 1.0329]$
δ_1	0.8000	0.5000	0.3023	0.1672	$[0.0551, 0.6069]$
q_D	0.9859	0.0139	0.9919	0.0060	$[0.9805, 0.9986]$
DIC	452.79				

Appendix C. Figures

Figure C.1. Model III: Hamilton Model with a Bounce-Back Effect

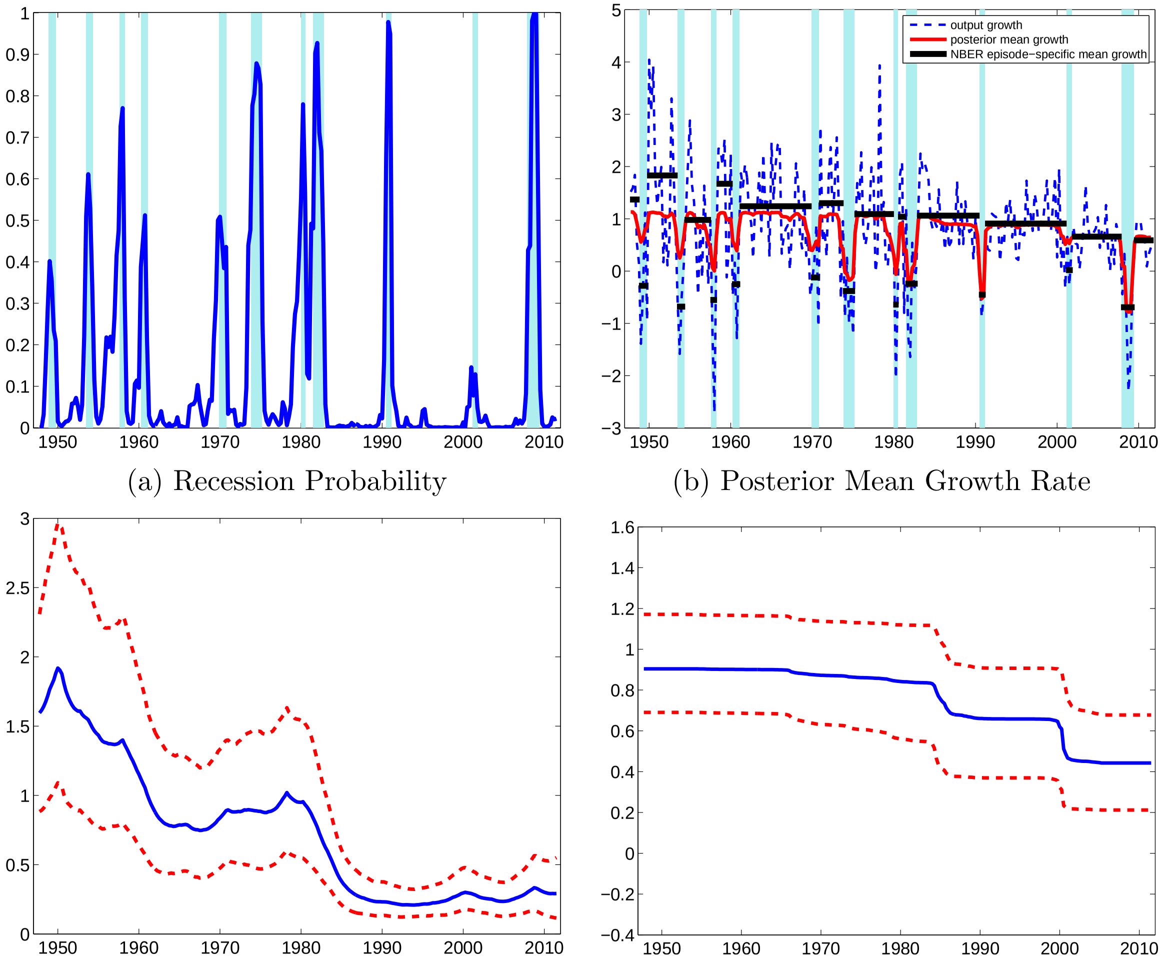


(a) Stochastic Volatility $\sigma^2_{e,t}$



(b) Long-run Growth Rate δ_t

Figure C.2. Model IV: Hamilton Model with a Structural Break in δ



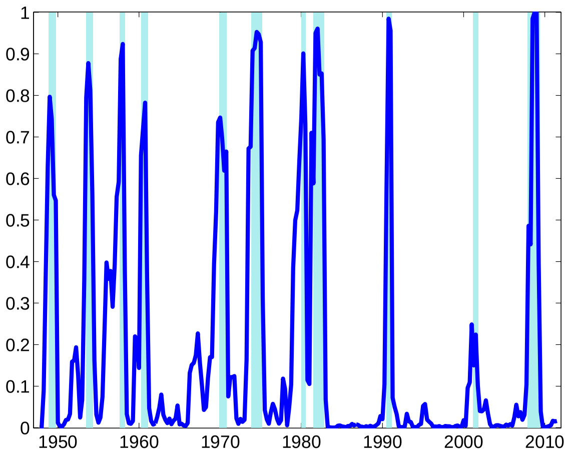
(a) Recession Probability

(b) Posterior Mean Growth Rate

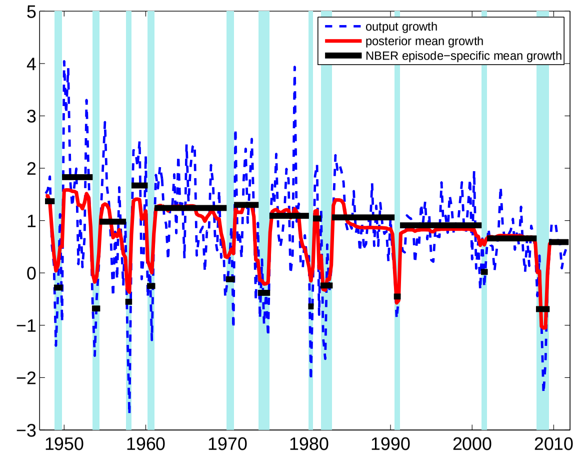
(c) Stochastic Volatility $\sigma^2_{e,t}$

(d) Long-run Growth Rate δ_t

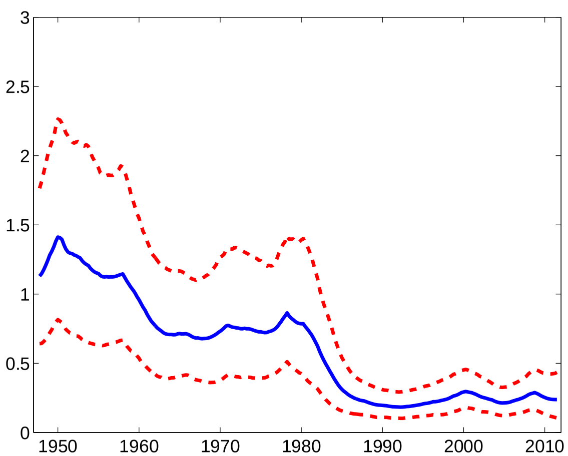
Figure C.3. Model V: Proposed Model with a Structural Break in δ



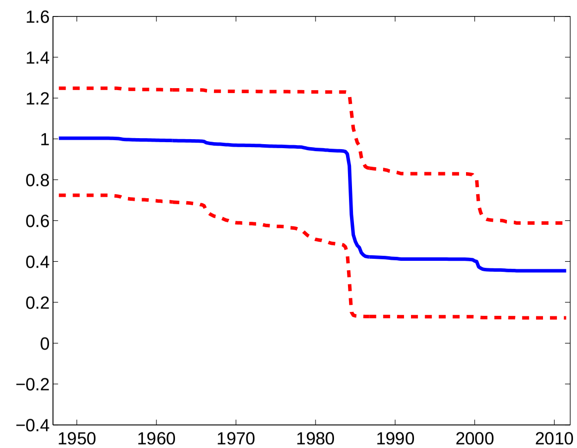
(a) Recession Probability



(b) Posterior Mean Growth Rate

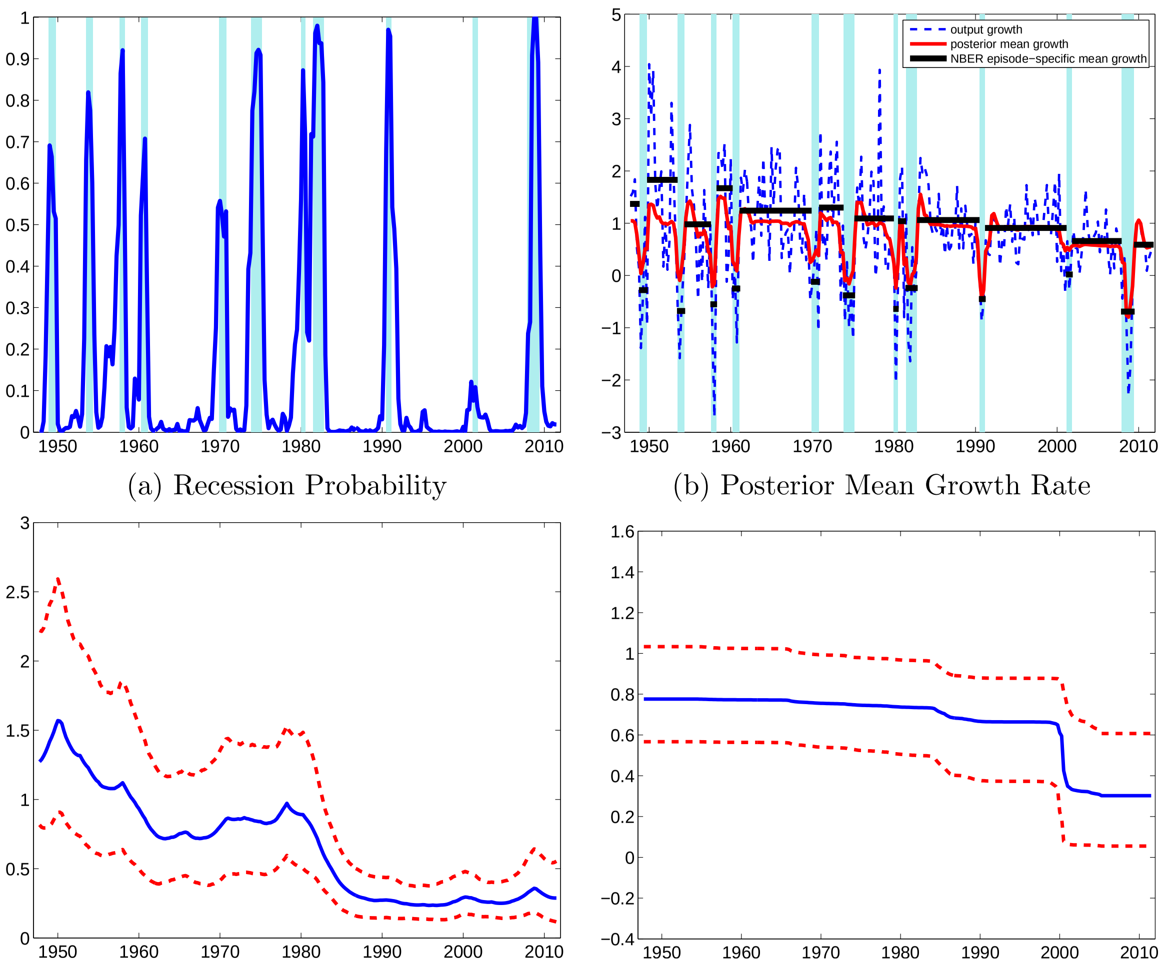


(c) Stochastic Volatility $\sigma^2_{e,t}$



(d) Long-run Growth Rate δ_t

Figure C.4. Model VI: Hamilton Model with a Bounce-Back Effect and a Structural Break in δ



(a) Recession Probability

(b) Posterior Mean Growth Rate

(c) Stochastic Volatility $\sigma^2_{e,t}$

(d) Long-run Growth Rate δ_t