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Treasure Game

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Treasure game*

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Abstract

A prize is located at an unknown point on an island. In each period, each of n players searches a subset of the as yet unsearched portion of the island. If one player alone finds the prize he wins it and the game ends. Players have a per-period discount factor and a search cost proportional to area searched. Efficient symmetric Markov perfect equilibria are characterized when search is observable. Equilibria for $n \geq 2$ exhibit two types of inefficiency: a tragedy of the commons (for small islands) and free riding (for large islands). For $n \geq 3$, equilibrium properties are non-monotonic: players may be better off searching larger islands, and larger islands may take less time to search. When search is unobservable and players are sufficiently impatient, multi-player search can be efficient. The model is very general: applications include R&D races, team production, and extraction of exhaustible resources.

Keywords: R&D, search, uncertainty.
JEL classifications: O32.

1 Introduction

Consider pharmaceutical firms participating in an R&D race for drug discovery, or paparazzi looking for a movie star in city hotels, or researchers looking for solutions to the six Millennium Prize Problems in mathematics. All these situations are examples of a common general problem. Namely, a treasure hunt where the prize value is common knowledge, but the search costs are unknown ex ante. This is the problem we consider in this paper.

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In particular, we analyze a dynamic model in which a given number of players search for a treasure hidden somewhere on an island of a given area. The value of the treasure is common knowledge, and search is costly. Once the treasure is found the game ends. In each period, all players make their search decisions simultaneously. If the treasure is not found in the current period, search in the next period occurs over the remaining unsearched area. If several players find the treasure simultaneously (search the same part of the island), each of them incurs their costs, but the treasure will be destroyed (players do not get any treasure).[1] We consider the case when search is observable; that is, during the search players are informed about the areas that have already been searched by their opponents.[2]

We analyze a game in which each state is described by the remaining unsearched area. There are multiple subgame perfect equilibria in the game; we restrict our attention to the symmetric Markov perfect equilibria (SMPE).[3] Among all SMPE, we only consider the efficient SMPE; that is, the equilibrium with the highest total expected payoff in the absence of collusion. We find and completely characterize the unique efficient SMPE when there are one or two players. For more than two players we can characterize the unique efficient SMPE when the maximum number of search periods is two.

We compare the efficient SMPE for multi-player search to the case of monopoly. Relative to the latter, multi-player search is typically inefficient except for very small islands when players behave as a cartel and search lasts just one period. In general, there are two types of potential inefficiency. First, in the case of small islands, multiple players search too fast; the probability of finding a treasure is relatively high, which means players have an incentive to over-search in the current period. This is a standard *tragedy of the commons* effect. It leads to over-investment in comparison with the case of monopoly. Second, in the case of large islands, players undertake insufficient search; the probability of finding the treasure is relatively low, so the immediate payoff from search is negative. Players want others to search and incur current losses, hoping that the treasure will not be found in the current period. In other words, there is an incentive to postpone search to a future period, when it will be more profitable. This is a standard *free-riding* effect. It leads to under-investment in comparison with the monopolist. Note that in the present model, in contrast to the existing literature, both the tragedy of the commons and the free-riding effect may endogenously arise within the same project.

Since search is costly, it seems natural to conjecture that a smaller island (lower

[1]This assumption is standard in the R&D literature. Intuitively, if several players discover the treasure simultaneously, fierce competition between them runs down the surplus to zero. A good example of such a situation for just two players is Lockheed and Douglas jet development in the 1960s. For more detail, see *The Economist (1985)*; and Chatterjee and Evans (2004). Many examples of simultaneous discoveries in science can be found in Merton (1973).

[2]We also consider the case with unobservable search in Section 7.

[3]Imposing Markov perfection not only makes our analysis simpler, while still being consistent with rationality, but it also makes our results directly comparable to those in the previous literature. See Maskin and Tirole (1988), Bhaskar et al. (2010) and Battaglini et al. (2012) for a general discussion of why the use of SMPE is appropriate.

search costs) is better than a bigger island for all players. In fact, it turns out not to be the case. By example, we illustrate that players might be worse off with a smaller island. This surprising observation, which we refer to as Puzzle 1, means that an increase in expected costs might make all the players better off. Puzzle 1 has the following intuitive explanation. If the island is small, the tragedy of the commons effect is strong, and players oversearch the island. If the island area is increased, the tragedy of the commons effect decreases, and players search the island more efficiently. It turns out that with three or more players this efficiency improvement may be large enough to outweigh the increase in the cost of searching the larger island.

It also seems natural to anticipate that the expected number of search periods monotonically increases with the island area. As we show in the Example this conjecture is also incorrect. In fact, a larger island can either speed up or slow down the search process. We refer to this observation as Puzzle 2. This puzzle can be explained by inefficient player behavior when the island is relatively large and there are multiple players. Due to the *free-riding* effect, players have a greater incentive to underinvest. When there are many players, this effect may be very strong, leading to non monotonicity of the search function. In the extreme case this may lead to non existence of the efficient SMPE.

In the special case when the cost of searching the entire unsearched area of the island is exactly equal to the treasure value, the tragedy of the commons and free-riding effects are absent, and multi-player search can reproduce the monopoly outcome. This happens when the discount factor is sufficiently low, guaranteeing that players search the island in at most two periods. For this unique island area, players get zero expected payoff in period one. Consequently the objective function of players is to maximize their expected payoffs from the second period only. In the second period in the symmetric equilibrium, each firm receives a payoff proportional to the payoff of the monopoly. This guarantees that multi-player search reproduces the monopoly outcome.

There are two alternative interpretations of our model with observable search. First, as a dynamic game of financing a private good with an uncertain threshold. Barbieri and Malueg (2010) introduce threshold uncertainty into a private-values one-period model of voluntary provision of a discrete public good. In contrast to their model, we consider a dynamic game where a private good rather than a public good is being financed. Second, the reduced form generated in our model resembles the problem of extraction of exhaustable resources under common access or, equivalently, a multi-player cake-eating problem.[4]

We also consider the case when search is unobservable. Our main finding is that, if players are sufficiently impatient, multi-player search can reproduce the monopoly outcome. This is the case despite the fact that an arbitrarily large number of players participate in a non-cooperative dynamic game and history-dependent strategies can not be relied upon to support cooperation. The result can be explained by the fact

[4]See for example Long (2011) for a recent survey of dynamics games in the economics of natural resources.

that neither the *tragedy of the commons* nor *free-riding* effect are possible when search is unobservable.

In contrast to much of the existing literature, we focus on the dynamics of investment in relation to private goods. Previous work typically investigates dynamic models of investment in relation to contributions to public goods.[5] Bonatti and Hörner (2011) is the work that is closest in spirit to our paper. In their framework, two agents contribute to a public project that can be either good or bad. At any time if the investment is unsuccessful, the probability of the project being good decreases. In contrast, in our model unsuccessful search leads to a decrease in unsearched area and therefore brings the successful conclusion of the project potentially closer. The main findings of their paper are that: agents always underinvest; deadlines can partly mitigate this under-investment problem; and delay is greater when efforts are observable rather than unobservable. In our model both under-investment and over-investment are possible, and deadlines are not beneficial because the private-good nature of our problem implies that the free-riding typically associated with public goods does not arise. In addition, in our setting when players are sufficiently impatient, they can behave efficiently if their search efforts are unobservable.

Previous work on private goods mostly deals with situations that are either static or involve complete information.[6] The typical outcome of these models is that firms overinvest. One of the few papers that considers the dynamics of investment is Reinganum (1981), who shows that in a dynamic R&D race where each firm chooses a time path of expenditures, firms may underinvest as compared to the monopoly outcome. Aggregate expenditure on R&D may, therefore, depending on the exogenous parameters, be either too high or too low relative to the monopoly outcome. To simplify the analysis, Reinganum assumes that the success function is exponential. As a result, previously acquired knowledge does not change the probability of current success in the race; that is, the equilibrium strategies may be time-independent. There are many situations, however, where the memorylessness assumption is not satisfactory; for example, when the search domain, while potentially large, is finite. Consideration of this case permits us to shed light on the dynamics of when over-investment or under-investment are likely to arise.[7]

Chatterjee and Evans (2004) analyze a R&D race, where two competing firms simultaneously choose between two research projects, where investment is observable and it is common knowledge that exactly one of these projects will be successful if enough investment is made. While agents in their model decide which area to search (how much they search each period is exogenously determined), agents in our model decide how much to search (the location has no importance).

[5]See, for example, Admati and Perry (1991), Marx and Matthews (2000), Lockwood and Thomas (2002), Compte and Jehiel (2004), Matthews (2012) and Battaglini et al. (2012).

[6]See, for example, Loury (1979), Dasgupta and Stiglitz (1980a, b) and Lee and Wilde (1980), and Reinganum (1989) for a literature survey.

[7]Doraszelski (2003) investigates the parallel question of when the firm that is behind in the race engages in catch-up behavior. Using simulation, he obtains richer investment dynamics by discarding the memorylessness assumption. In this paper we derive analytical results in the same vein.

Our paper is related to the literature on strategic experimentation with publicly observable actions and outcomes.[8] This strand of literature uses the two-armed bandit framework to model the trade-off between experimentation and exploitation in teams. In particular, Klein and Rady (2011) assume a negative correlation of the quality of the risky arm across players. Note that strategic interaction in their model arises out of purely informational concerns. In our context, Klein and Rady assume that a player benefits from the other player's previous periods of unsuccessful search. However, contrary to us, there is no payoff rivalry among players. In their model all Markov perfect equilibria are in cutoff strategies. In contrast, in our framework once players begin searching they stop only if they find the treasure.

Finally, there is a literature that models research contests as rank-order tournaments. In contrast to our paper, this literature analyzes the situation in which there are multiple potential innovations that compete against each other. Some examples recently discussed in the literature include: a 1992 refrigerator competition (see Taylor, 1995), a 1829 steam locomotion tournament (see Fullerton and McAfee, 1999), a 1714 British contest for a method of determining longitude at sea (see Che and Gale, 2003). As with the literature on strategic experimentation, and in contrast to our model, all Markov perfect equilibria in these papers are in cutoff strategies.

2 An Example

Consider the following two-period game. Player $i = 1, 2$ may search the island of size $x(0) = 1$ in two periods, $t = 1, 2$, in order to find a treasure of value 2. The cost of searching for either player is linear, $C(x) = x$, and there is an equal probability that the treasure will be located at any given point on the island. If players search the same area and find the treasure simultaneously, each of them incurs their costs, but the treasure will be destroyed. Payoffs from the second period are discounted at a common factor $\delta = 0.75$.

First, we consider the case when both players observe the search choices of the other player. Let us solve the game by backward induction. Consider the second-period subgame where the treasure is not found in the first period and the unsearched area of the island is $x \leq x(0)$. Note that if player i searches $I^i(2)$ in the second period and there is no duplication, then his expected second-period payoff is

$$2I^i(2)/x - I^i(2) = I^i(2)(2/x - 1).$$

Provided there is no duplication, each player has a dominant strategy to search as much as possible in the second period. Therefore in the unique symmetric SPE, $I^1(2) = I^2(2) = x/2$ and the value for player i in the subgame with the unsearched area of x is

$$V_1(x) = 1 - x/2.$$

[8]See, for example, Bolton and Harris (1999), Keller, Rady and Cripps (2005), Keller and Rady (2010) and Klein and Rady (2011).

Now consider the first period. Note that if player i searches $I^i(1)$ in the first period, then his expected payoff from two-period game is

$$2I^i(1)/x(0) - I^i(1) + \delta(x(0) - I^1(1) - I^2(1))V_1(x(0) - I^1(1) - I^2(1)),$$

where player i incurs searching cost of $I^i(1)$, gets immediate expected payoff of $2I^i(1)/x(0)$, and anticipates the expected payoff of $\delta(x(0) - I^1(1) - I^2(1))V_1(x(0) - I^1(1) - I^2(1))$ in the second period. Simplifying the above expression yields

$$I^i(1) + \frac{\delta}{2}(1 - I^1(1) - I^2(1))(1 + (I^1(1) + I^2(1))).$$

The first-order conditions with respect to $I^i(1)$ are

$$1 - \delta(I^1(1) + I^2(1)) = 0.$$

Whenever $I^1(1) + I^2(1) \leq x(0) = 1$ and provided there is no duplication, player i has incentives to search as much as possible in the first period. Consequently, there is a unique symmetric SPE that involves each player finishing the search in one period by searching a complementary half of the island; that is, $I^1(1) = I^2(1) = x(0)/2 = 1/2$. Player i's value in the two-period game with the island size of $x(0) = 1$ is $V_2(1) = \frac{1}{2}$.

Next, consider the case when only one player (a monopolist) searches the island in two periods. Similar to the argument above, one can define the monopolist's value in the second-period subgame with an unsearched area of x as

$$V_1(x) = 2 - x.$$

Consider the first period now. If the monopolist searches $I(1)$ in the first period, his expected payoff from the two-period game is

$$2I(1)/x(0) - I(1) + \delta(x(0) - I(1))V_1(x(0) - I(1)).$$

Simplifying the above expression yields

$$I(1) + \delta(1 - I(1))(1 + I(1)) = I(1) + \delta(1 - (I(1))^2).$$

The first-order conditions with respect to $I(1)$ give

$$1 - 2\delta I(1) = 0.$$

Consequently, there is a unique SPE that involves searching 2/3 of the island in the first period and the remaining 1/3 of the island in the second period if necessary; that is, $I(1) = 2/3$ and $I(2) = 1/3$. The value for the monopolist of the two-period game with the island size of $x(0) = 1$ is $V_2(1) = \frac{13}{12}$.

Finally, we consider the case when players do not observe the search choices of the other player. Specifically, we assume that each player searches in two periods without observing what area has been searched by the other player. To make this case consistent with the observable case, it is assumed that before any search is undertaken, players coordinate on how to divide the island in two equal parts. Then the strategy of each player is what portion of his own land, and the other player's land, to search each period. Note that if a player searches the area associated with the other player, there could be some duplication.

Let us show that the efficient equilibrium derived earlier could be supported as a Nash equilibrium when the search choices are unobservable; that is, each player searches 2/3 of his own area in the first period and 1/3 of his own area in the second period (if necessary), and neither player searches the area associated with the other player. First, given that in the proposed equilibrium both players search efficiently, they have no incentive to deviate by changing the division of search between the first and the second periods of their own portion of the island – this is the optimal search strategy they could adopt. Second, let us show that players also have no incentive to search the area associated with the other player. A player can deviate from the equilibrium described, by searching the other player's area either in the first or in the second period. Given that in the second period the other player finishes the search in his area, searching the area of the other player in the second period is clearly disadvantageous. On the other hand, searching the other player's area in the first period by ϵ would generate an additional payoff of

$$-\epsilon + 2\epsilon(1 - 2/3) = -\epsilon/3.$$

The first term is the cost of searching the other player's area, while the second term is the expected benefit to the deviator who searches ϵ, while the 'incumbent' searches 2/3 of his area. Given that the search by the 'incumbent' is unobservable, it is assumed that from the deviator's prospective, the 'incumbent' searches his area randomly. As one can see the combined effect is negative for any positive ϵ. Consequently neither player has an incentive to deviate by searching the other player's area.

Thus, when the search is observable, players search too fast. This is what we refer to as the *tragedy of the commons* effect. The total value of the game for two players is $W_2 = 2V_2 = 1$. On the other hand, when search is unobservable, players search efficiently and $W_2 = 2V_2 = \frac{13}{12}$. Note that these results are consistent with the findings of Bonatti and Hörner (2011).

3 The Model

There are $n \geq 1$ players. They are looking for a treasure which is hidden somewhere on an island. The treasure has the same value $R > 0$ for all players, and there is an equal probability that the treasure will be located at any given point on the island.[9] Denote the area player i searches in period t as $I^i(t)$. Let the unsearched area of the island at period t be given by $x(t)$. The size of the island is assumed to be $x(0) > 0$.

[9] We focus our attention on uniform distribution because this is the most realistic assumption when there is no information about the island. See Section 8 for further discussion.

At period $t = 0$ all players simultaneously choose how much to search for the treasure. The search is costly. If player i searches $I^i(0)$, his search cost is $-I^i(0)$. It is assumed that if players search the same area and find the treasure simultaneously, each of them incurs costs, but the treasure will be destroyed.[10] Consequently, in equilibrium players never search the same area. All players together search

$$J(0) = I^1(0) + ... + I^n(0).$$

Note that given players never search the same area and searching outside the island has no benefit, it must be the case that $J(0) \leq x(0)$.

If $J(0) = x(0)$, player i has a $\frac{I^i(0)}{x(0)}$ probability of finding the treasure, and the game ends. Player i obtains the following expected payoff:

$$\frac{I^i(0)}{x(0)}R - I^i(0).$$

If $J(0) < x(0)$, player i has a $\frac{I^i(0)}{x(0)}$ probability of finding the treasure, and the game ends with probability $\frac{J(0)}{x(0)}$. If the treasure is not found at period $t = 0$ (this happens with probability $1-\frac{J(0)}{x(0)}$), the unsearched area of the island shrinks to $x(1) = x(0) - J(0)$, and the game proceeds to the next period $t = 1$.

At period $t > 0$, each player knows the *history* $h(t) = (x(0); J(0), ..., J(t-1))$ and all players simultaneously choose how much to search for the treasure on the previously unsearched area of size $x(t)$. Given that players never search the same area and searching the previously searched area has no benefit, it must be the case that $J(t) \leq x(t)$.

If $J(t) = x(t)$, player i has a $\frac{I^i(t)}{x(t)}$ probability of finding the treasure. The expected payoff for player i is

$$\delta^t \frac{I^i(t)}{x(t)}R - (I^i(0) + \delta I^i(1) + ... + \delta^t I^i(t)),$$

where δ is the common discount factor, and the game ends.

If $J(t) < x(t)$, player i has a $\frac{I^i(t)}{x(t)}$ probability of finding the treasure, and the game ends with probability $\frac{J(t)}{x(t)}$. If the treasure is not found at period t (this happens with probability $1 - \frac{J(t)}{x(t)}$), the unsearched area of the island shrinks to

$$x(t+1) = x(t) - J(t).$$

The new unsearched area is equal to the previous one, minus the searched part.

We assume that each player observes how much the other players have searched previously before making his search plans.[11] Note that all search costs are sunk, but

[10] An alternative assumption is discussed in Section 8.

[11] Section 7 considers the case when search is unobservable.

only one player (if any) can find the treasure. Moreover, the value of the prize is known from the beginning, but the search costs for each player will be determined only at the end of the game.

Player i's strategy is an infinite sequence of functions specifying how much to search each period contingent upon any possible sequence of previous searches. The game we consider is stochastic, and any history can be summarized by the "state", the current unsearched area. We will consider only Markov strategies in which the past influences the current play only through its effect on the current unsearched area. A pure Markov strategy for player i is a time-invariant map $I^i : X \to X$, where $X = [0, x(0)]$. We also will restrict our attention only to symmetric equilibria. Therefore, the solution concept is a *symmetric Markov perfect equilibrium* (SMPE).[12] Moreover, because in general there exist multiple SMPE, we focus on the efficient SMPE; that is, the equilibrium with the highest total expected payoff in the absence of collusion.[13]

We use the following approach to obtaining the efficient SMPE. Player i takes the state-contingent investment plans of his rivals as given; these plans can be aggregated to obtain $I^{-i}(x)$. Given this function, player i solves a standard optimization problem and chooses his optimal investment, $I^i(x)$. However, given that the function $I^{-i}(x)$ is endogenous to the model, to obtain the symmetric equilibrium we need to find the function $I^{-i}(x)$ such that $(n-1)I^i(x) \equiv I^{-i}(x)$, where $I^i(x)$ is the optimal investment for player i when he takes I^{-i} as given.

Player i solves the following Bellman equation:

$$V(x) = \max_{0 \le I^i \le x - I^{-i}} -I^i + \frac{I^i}{x} R + \delta \left(1 - \frac{I^i + I^{-i}}{x}\right) V(x - I^i - I^{-i}), \tag{1}$$

where x is the part of the island which is still unsearched before the current period, $V(x)$ is the value function for each player (we use the symmetry assumption here) and the solution has to satisfy $(n-1)I^i = I^{-i}$. The first term in equation (1) describes the player's costs of search in the current period. The second term is the player's expected value from finding the treasure in the current period. The last term is the player's expected value from future periods.

Define that part of x which player i does not search in the current period by

$$y = x - I^i$$

and the part of x no player searches in the current period by

$$z = x - (I^1 + ... + I^n) = y - I^{-i}.$$

[12] We discuss both subgame perfect and asymmetric Markov perfect equilibria in Section 8.

[13] If two efficient SMPE have the same total expected payoff, it is assumed without loss of generality that the efficient SMPE with the largest number of potential search periods is selected. This criterion guarantees that the efficient SMPE is unique.

Equation (1) can be rewritten in the following way

$$V(x) = \max_{I^{-i} \leq y \leq x} \{-(x-y) + R(x-y)/x + \delta z V(z)/x\}. \tag{2}$$

Note that x, y, z, R and $V(x)$ are of the same unit measure. For convenience, we assume without loss of generality that $R = 1$ to work with unit free variables. Equation (2) transforms into

$$V(x) = \max_{I^{-i} \leq y \leq x} \{-(x-y) + (x-y)/x + \delta z V(z)/x\}. \tag{3}$$

To simplify the exposition, it is convenient to introduce the following function:

$$\Psi(x) \equiv xV(x). \tag{4}$$

From definition (4), it follows that

$$\Psi(x) \geq 0 \text{ for any } x.$$

Equation (3) in terms of $\Psi(x)$ can be rewritten as

$$\Psi(x) = \max_{(n-1)I \leq y \leq x} \{(1-x)(x-y) + \delta\Psi(z)\}, \tag{5}$$

and the solution has to satisfy $n(x-y) = x - z$, where for simplicity $I = I^i$.

4 Alternative Interpretations

In this section we introduce two alternative interpretations of the model with observable search.

4.1 Private good provision with uncertain threshold

There are $n \geq 1$ players, who want to finance a private good as a team. The threshold of investment at which the good of value R becomes available is not known, but the players know that the threshold is distributed uniformly from 0 to some value x. In each period, all players simultaneously choose how much to contribute. If player i contributes I^i, his cost is I^i. All players together contribute

$$J = I^1 + \ldots + I^n.$$

Note that since investing more than x has no benefit but is costly, it must be the case that $J \leq x$. If $J < x$, players have a J/x probability of financing the private good in the current period. If they are successful they share value R in proportion to their contributions; that is, player i gets I_i/J of the good (or gets R with probability I_i/J). If they are unsuccessful, the game is repeated next period with the threshold distributed uniformly from 0 to $x - J$, where future payoffs are discounted by the common discount factor δ. If $J = x$, players finance the private good in the current period. They share value R in proportion to their contributions, so that player i gets I_i/J of the good. One can see that the Bellman equation (1) describes this game. To our knowledge, our model is the first to consider a dynamic process of team investment in which the total cost to complete the project is uncertain.

4.2 Extraction of exhaustable common-pool resources

There are $n \geq 1$ firms, who each period choose simultaneously how much of a non-renewable resource to extract. The resource endowment is given by x and all firms have common access to the resource. If firm i extracts I^i in the current period, it derives net utility of $U_i = (1 - x)I^i$ from selling the resource in the market. Firm i maximizes a discounted sum of utilities, where δ is the common discount factor. It is evident that the Bellman equation (5) describes this game. Note that when $x > 1$, net utility from selling the resource in the market is negative; this assumption diverges from the current literature. Our setup leads to some interesting results, non-monotonicity of both the value and search functions for example, that are new to the existing literature on the economics of natural resources.

5 Analysis of the Model

In this section we derive a general procedure for finding the efficient SMPE. First, to rule out cases where no investment is made in equilibrium we introduce the following definition.

Definition 1. *A SMPE is trivial if $I(t) = 0 \ \forall \ t$.*

We want to focus our attention on SMPE that are non-trivial. Before outlining the first lemma, we introduce the following operator B on the set of continuous functions

$$(B\Psi)(x) \equiv \max_{(n-1)I \leq y \leq x} \{(1 - x)(x - y) + \delta\Psi(z)\}. \tag{6}$$

Lemma 1.
(a) Any non-trivial SMPE involves $I(t) > 0$ for $0 \leq t \leq T$, and $I(t) = 0$ for $t > T$, where T is finite.
(b) All SMPE can be obtained in a finite number of steps applying the following sequence $\{\Psi_k\}$, where

$$\Psi_0 \equiv 0, \qquad \Psi_k \equiv B\Psi_{k-1}, \quad k = 1, 2, \ldots$$

Proof. See the Appendix.

Note that with the help of Lemma 1, one can construct the sequence $\{\Psi_k\}$ and find all SMPE. This is called the value-iteration procedure. This procedure is equivalent to using backward induction. In general, there will be multiple SMPE. For simplicity we focus on the unique efficient SMPE.

5.1 Construction of Ψ_1 and V_1

Let us start from the end of the search process. What will be the value of the game if players could only search for at most one period? Equation (5) transforms into

$$\Psi_1(x) = \max_{(n-1)I \le y \le x} \{(1-x)(x-y)\}. \tag{7}$$

It is evident that the optimal y can be described in the following way[14]:

$$y = \begin{cases} x, & \text{if} \quad x > 1, \\ (n-1)I, & \text{if} \quad x \le 1. \end{cases}$$

If $x \le 1$, then in SMPE players search the whole island, $I^1 + ... + I^n = nI = x$. Consequently,

$$y = \frac{(n-1)x}{n} \quad \text{and} \quad z = 0 \ \text{ if } \ x \le 1.$$

Therefore the solution of (7) is

$$\Psi_1(x) = \begin{cases} P_1(x), & \text{if} \quad x \le u_1 = 1, \\ 0, & \text{if} \quad x > u_1 = 1, \end{cases} \tag{8}$$

where $u_1 = 1$ is the largest positive root of polynomial $P_1(x) = x(1-x)/n$. For future reference note that

$$P_1(x) = \frac{a_1}{n}(1-x)^2 + \frac{b_1}{n}(1-x) + \frac{c_1}{n}, \tag{9}$$

where

$$a_1 = -1, \quad b_1 = 1, \quad c_1 = 0.$$

If players can search the island for at most one period, then the only SMPE is $(I(x), ..., I(x))$, where

$$I(x) = \begin{cases} \frac{x}{n}, & \text{if} \quad x \le u_1 = 1, \\ 0, & \text{if} \quad x > u_1 = 1. \end{cases}$$

Define the value of the game for each player (if the players can search the island for at most k periods) as $V_k(x) \equiv \Psi_k(x)/x$, for any $x \ge 0$. From the above definition, it follows that

$$V_1(x) = \begin{cases} (1-x)/n, & \text{if} \quad x \le u_1 = 1, \\ 0, & \text{if} \quad x > u_1 = 1. \end{cases}$$

[14] Note that if $x = 1$, then *any* $y \in [(n-1)I, x]$ is optimal. We assume that players choose $y = (n-1)I$ in this case.

5.2 Construction of Ψ_2 and V_2

What will be the value of the game if players can search the island for at most two periods? In general there could be three possibilities, depending on the island size. The first possibility is that the players search the island for just one period. Intuitively this happens for small values of x because it is too costly to wait for another period when the island is very small. The second possibility is that the players search the island for at most two periods. This happens for middle values of x. Finally, players can find search to be too costly, and abstain from searching at all. This happens when the initial island is large (costs are very high).

We have already considered the first possibility in the previous section. Now we analyze the situation when players plan to search for at most two periods. The first step is to construct $\Psi_2(x)$. Equation (5) in this case transforms into

$$\Psi_2(x) = \max_{(n-1)I \le y \le x} \{(1-x)(x-y) + \delta \Psi_1(z)\}. \tag{10}$$

The necessary condition for y to be optimal in the interior of $[0, x]$ is

$$-(1-x) + \delta \Psi_1'(z) = 0. \tag{11}$$

In order to continue the search for the second period, the remaining island size has to satisfy

$$0 \le z \le u_1. \tag{12}$$

The sufficient condition for y to be optimal in the interior of $[0, x]$ is satisfied because

$$\Psi_1''(z) = a_1 < 0.$$

The way to proceed is to construct the equilibrium with the help of condition (11), and then show that the derived equilibrium satisfies condition (12).

From expressions (11) and (8), it follows that

$$-(1-x) + \delta \left(\frac{1-2z}{n} \right) = 0.$$

Consequently,

$$z(x) = \frac{n(x-1) + \delta}{2\delta}. \tag{13}$$

It is now straightforward to show that

$$y = \frac{(n-1)x + z(x)}{n} = \frac{2\delta(n-1)x + n(x-1) + \delta}{2\delta n}. \tag{14}$$

Substituting (14) and (13) into equation (10), we obtain a *spline* of degree two on the interval $[0, u_2]$[15]:

$$\Psi_2(x) = \begin{cases} \Psi_1(x), & \text{if} \quad 0 \le x \le t_1, \\ P_2(x), & \text{if} \quad t_1 < x \le u_2, \\ 0, & \text{if} \quad x > u_2, \end{cases} \tag{15}$$

where $u_2 > 0$ is the largest positive root of polynomial

$$P_2(x) = \frac{a_2}{n}(1-x)^2 + \frac{b_2}{n}(1-x) + \frac{c_2}{n}, \tag{16}$$

with

$$a_2 = -1 - s, \quad b_2 = \frac{1}{2}, \quad c_2 = \frac{\delta}{4},$$

and

$$s = \frac{n(n-2)}{4\delta}. \tag{17}$$

In order to find u_2, we need to solve the quadratic equation $P_2(u_2) = 0$. It is easy to check that

$$u_2 = 1 + \frac{\sqrt{4\delta(s+1)+1} - 1}{4(s+1)}.$$

The point $x = t_1$ is the first knot of the spline. When $x = t_1$ players are indifferent between searching the island for two periods or for just one period:

$$\Psi_1(t_1) = \Psi_2(t_1). \tag{18}$$

From (9) and (16), we get[16]

$$t_1 = 1 - \frac{\delta}{n}. \tag{19}$$

All our calculations so far are valid for any $n \ge 1$. Consider parameter s now. From expression (17), it follows that

$$s \begin{cases} < 0, & \text{if} \quad n = 1, \\ = 0, & \text{if} \quad n = 2, \\ > 0, & \text{if} \quad n \ge 3. \end{cases} \tag{20}$$

Condition (20) characterizes three different types of behavior in SMPE. There are three cases: $n = 1$ (a monopoly); $n = 2$ (a duopoly); and $n \ge 3$ (an oligopoly).

It is straightforward to check that the solution given by (15) satisfies condition (12) for any $x \in [t_1, u_2]$. Therefore, if the players can search the island for at most

[15] A spline is a special function defined piecewise by polynomials, see for example Ahlberg, Nielson, and Walsh (1967).

[16] Condition (18) can have one or two solutions: one is given by (19), the other one exists only if $n > 2$ and is equal to $t_1 = 1 + \frac{\delta}{n-2}$. If $n = 1$, both solutions to (18) coincide. If $n = 2$, there is only one solution, which is given by (19). Finally if $n \ge 3$, it is easy to see that $\Psi_1(1 + \frac{\delta}{n-2}) < 0$, which means that only (19) is relevant.

two periods, then $y(x)$ is a spline of degree one on the interval $[0, u_2]$ with one knot $x = t_1$:

$$y(x) = \begin{cases} \frac{(n-1)x}{n}, & \text{if} \quad x \le t_1, \\ \frac{2\delta(n-1)x+n(x-1)+\delta}{2\delta n}, & \text{if} \quad t_1 < x \le u_2, \\ x, & \text{if} \quad x > u_2. \end{cases}$$

The efficient SMPE (if the players can search the island for at most 2 periods) is also a spline of degree one on the interval $[0, u_2]$ with one knot $x = t_1$:

$$I(x) = \begin{cases} \frac{x}{n}, & \text{if} \quad x \le t_1, \\ \frac{(2\delta-n)x+n-\delta}{2\delta n}, & \text{if} \quad t_1 < x \le u_2, \\ 0, & \text{if} \quad x > u_2, \end{cases} \tag{21}$$

and the value function is

$$V_2(x) = \begin{cases} V_1(x), & \text{if} \quad x \le t_1, \\ P_2(x)/x, & \text{if} \quad t_1 < x \le u_2, \\ 0, & \text{if} \quad x > u_2. \end{cases}$$

We can describe the construction of Ψ_k and V_k now. In the next subsection we obtain a complete characterization of the functions Ψ_k, V_k and derive the efficient SMPE in the cases of $n = 1$ and $n = 2$. If $n \ge 3$, then finding the efficient SMPE is not always feasible, which we illustrate in Section 6.

5.3 Construction of Ψ_k and V_k when $n = 1$ and $n = 2$

What will be the value of the game if players can search the whole island for at most $k \ge 3$ periods? In general there could be $k + 1$ possibilities, depending on the island size $x(0)$. The players can plan to search the island for at most $1, 2, ..., k$ periods, or not to search at all.

First, let us construct $\Psi_k(x)$. Equation (5) in this case transforms into

$$\Psi_k(x) = \max_{(n-1)I \le y \le x} \{(1 - x)(x - y) + \delta\Psi_{k-1}(z)\}. \tag{22}$$

A necessary condition for y to be the optimal value in the interior of $[0, x]$ is

$$(1 - x) = \delta\Psi'_{k-1}(z). \tag{23}$$

In order to continue search for the next period, the new value of x has to satisfy

$$t_{k-2} \le z \le u_{k-1}, \tag{24}$$

where $t_0 = 0$. The sufficient condition for y to be the optimal value in the interior of $[0, x]$ is satisfied if

$$\Psi''_{k-1}(z) < 0. \tag{25}$$

We will use condition (23) to find z, and then show that it satisfies conditions

(24) and (25). Note that if function $\Psi_{k-1}(x)$ in (22) is a quadratic polynomial, then $\Psi_k(x) = B\Psi_{k-1}(x)$ has to be a quadratic polynomial as well. Since $P_1(x)$ and $P_2(x)$ are quadratic polynomials by (9) and (16), any $P_k(x)$ can be represented in the following form:

$$P_k(x) = \frac{a_k}{n}(1-x)^2 + \frac{b_k}{n}(1-x) + \frac{c_k}{n}, \quad k \geq 1. \tag{26}$$

From condition (23) and expression (26), it follows that

$$z(x) = 1 + \frac{\delta b_{k-1} + (1-x)n}{2\delta a_{k-1}}. \tag{27}$$

It is now straightforward to show that

$$y = x + \frac{z-x}{n} = x + \frac{(1-x)(n+2\delta a_{k-1})}{2\delta n a_{k-1}} + \frac{b_{k-1}}{2na_{k-1}}. \tag{28}$$

Hence

$$\Psi_k(x) = -(1-x)\left(\frac{(1-x)(n+2\delta a_{k-1})}{2\delta n a_{k-1}} + \frac{b_{k-1}}{2na_{k-1}}\right) + \delta\Psi_{k-1}(z).$$

Define the largest root of polynomial $P_k(x)$ as u_k, and that value of x such that players are indifferent between planning to search the area for k periods or for $k-1$ periods as knot t_{k-1}:

$$\Psi_{k-1}(t_{k-1}) = \Psi_k(t_{k-1}). \tag{29}$$

For the moment, let us assume that equation (29) has a unique solution. The uniqueness of the solution will be proved later in Lemma 2. Substituting (27) and (28) into equation (22), we obtain a *spline* of degree two on the interval $[0, u_k]$ with knots $t_1, ..., t_{k-1}$:

$$\Psi_k(x) = \begin{cases} \Psi_{k-1}(x), & \text{if} \quad 0 \leq x \leq t_{k-1}, \\ P_k(x), & \text{if} \quad t_{k-1} < x \leq u_k, \\ 0, & \text{if} \quad x > u_k, \end{cases} \tag{30}$$

where $P_k(x)$ is defined in (26). Therefore, if players plan to search the island for at most k periods, then $y(x)$ is a spline of degree one on the interval $[0, u_k]$ with knots $t_1, ..., t_{k-1}$:

$$y(x) = \begin{cases} \frac{(n-1)x}{n}, & \text{if} \quad x \leq t_1, \\ x + \frac{(1-x)(n+2\delta a_1)}{2\delta n a_1} + \frac{b_1}{2na_1}, & \text{if} \quad t_1 < x \leq t_2, \\ \vdots & \\ x + \frac{(1-x)(n+2\delta a_{k-2})}{2\delta n a_{k-2}} + \frac{b_{k-2}}{2na_{k-2}}, & \text{if} \quad t_{k-2} \leq x \leq t_{k-1}, \\ x + \frac{(1-x)(n+2\delta a_{k-1})}{2\delta n a_{k-1}} + \frac{b_{k-1}}{2na_{k-1}}, & \text{if} \quad t_{k-1} < x \leq u_k, \\ x, & \text{if} \quad x > u_k. \end{cases}$$

$I(x)$ is also a spline of degree one on the interval $[0, u_k]$ with knots $t_1, ..., t_{k-1}$:

$$I(x) = \begin{cases} \frac{x}{n}, & \text{if} \quad x \le t_1, \\ -\frac{(1-x)(n+2\delta a_1)}{2\delta n a_1} - \frac{b_1}{2na_1}, & \text{if} \quad t_1 < x \le t_2, \\ \vdots & \\ -\frac{(1-x)(n+2\delta a_{k-2})}{2\delta n a_{k-2}} - \frac{b_{k-2}}{2na_{k-2}}, & \text{if} \quad t_{k-2} \le x \le t_{k-1}, \\ -\frac{(1-x)(n+2\delta a_{k-1})}{2\delta n a_{k-1}} - \frac{b_{k-1}}{2na_{k-1}}, & \text{if} \quad t_{k-1} < x \le u_k, \\ 0, & \text{if} \quad x > u_k; \end{cases}$$

and the value function is

$$V_k(x) = \begin{cases} P_1(x)/x, & \text{if} \quad x \le t_1, \\ P_2(x)/x, & \text{if} \quad t_1 < x \le t_2, \\ \vdots & \\ P_{k-1}(x)/x, & \text{if} \quad t_{k-2} \le x \le t_{k-1}, \\ P_k(x)/x, & \text{if} \quad t_{k-1} < x \le u_k, \\ 0, & \text{if} \quad x > u_k, \end{cases}$$

or

$$V_k(x) = \begin{cases} V_{k-1}(x), & \text{if} \quad 0 \le x \le t_{k-1}, \\ P_k(x)/x, & \text{if} \quad t_{k-1} < x \le u_k, \\ 0, & \text{if} \quad x > u_k. \end{cases}$$

Let us now find a_k, b_k, and c_k for any $k \ge 2$. Using (22), (26) and (27), we get the following result[17]:

Theorem 1. *When $n = 1$ and $n = 2$*

$$a_k = -1 + \frac{s}{a_{k-1}}, \quad b_k = -\frac{b_{k-1}}{2a_{k-1}}, \quad c_k = \delta\left(c_{k-1} - \frac{b_{k-1}^2}{4a_{k-1}}\right), \quad k \ge 2, \tag{31}$$

where

$$s = \frac{n(n-2)}{4\delta},$$

and

$$a_1 = -1, \quad b_1 = 1, \quad c_1 = 0. \tag{32}$$

Theorem 1 describes all coefficients of the quadratic polynomials $P_k(x)$ when $n = 1$ and $n = 2$. Let us consider these two cases separately.

The following proposition characterizes the spline in (30) and the knots when $n = 1$.

[17] Note that if $a_{k-1} = 0$, the optimal sequence includes $k - 1$ steps only. This result follows from Lemma 2, and will be discussed at the end of the Lemma's proof.

Proposition 1. *When $n = 1$, system of difference equations (31) with initial conditions (32) has the following solution:*

$$a_k = -\frac{\sin(k+1)\varphi}{2v\sin k\varphi}, \quad b_k = \frac{v^{k-1}\sin\varphi}{\sin k\varphi}, \quad c_k = \frac{v^{2k-1}\sin(k-1)\varphi}{2\sin k\varphi}, \tag{33}$$

$$t_k = 1 - v^k\cos k\varphi, \quad u_k = 1 + \frac{v^k(\sin k\varphi - \sin\varphi)}{\sin(k+1)\varphi}, \quad k \geq 1, \tag{34}$$

where $v = \sqrt{\delta}$, and $\varphi = \arccos v$.[18]

Proof. See the Appendix.

The following proposition characterizes the spline in (30) and the knots when $n = 2$.

Proposition 2. *When $n = 2$, system of difference equations (31) with initial conditions (32) has the following solution*[19]*:*

$$a_k = -1, \quad b_k = \frac{1}{2^{k-1}}, \quad c_k = \left(\frac{(4\delta)^{k-1} - 1}{4^{k-1}(4\delta - 1)}\right)\delta, \tag{35}$$

$$t_k = 1 - \frac{3\delta + (4\delta)^k(\delta - 1)}{2^k(4\delta - 1)}, \quad u_k = 1 + \frac{1}{2^k}\left(\sqrt{\frac{1 - (4\delta)^k}{1 - 4\delta}} - 1\right), \quad k \geq 1. \tag{36}$$

Proof. See the Appendix.

Finally, we demonstrate that conditions (24), (25) hold, and equation (29) has a unique solution.

Lemma 2. *For $n = 1$ and $n = 2$ and any value of x, (30) satisfies conditions (24) and (25), and is the unique solution to (29).*

Proof. See the Appendix.

In order to complete the description of the efficient SMPE, we have to specify the maximum number of search periods for any value of x (i.e. we allow x to vary and find the maximum number of search periods). We will do that in the next subsection.

5.4 Maximum number of search periods when $n = 1$ and $n = 2$

In this subsection, we want to answer the following question: what is the minimum number k such that $V(x) \equiv V_k(x)$? In other words, what is the maximum number

[18] Note that $\sin k\varphi = 0$ implies $a_{k-1} = 0$ as in footnote 17.

[19] These expressions are presented in this concise form (rather than as sums) for expositional purposes. In particular, note that both numerators and denominators in c_k, t_k and u_k contain $(1 - 4\delta)$ term. When $\delta = 1/4$, these expressions are defined as their limits as $\delta \to 1/4$.

of periods (the worst case scenario) in which the treasure will be found for certain? Note that, in general, the answer to this question depends on the discount factor δ.

One way to answer the above question is to write the condition that the largest positive root of the quadratic polynomial $P_k(x)$ coincides with the largest positive root of the quadratic polynomial $P_{k+1}(x)$. This condition gives a critical value of δ; for slightly larger values of δ, there is an additional search period. This means that k is the smallest integer such that $\Psi_k(x) \equiv \Psi_{k+1}(x)$, or in other words $V(x) \equiv V_k(x)$. Since such k depends on δ, let us define for each n a knot discount factor $\delta_k(n)$ which is the solution to the following equation:

$$u_k(\delta_k(n)) = u_{k+1}(\delta_k(n)), \quad k \geq 2. \tag{37}$$

The knot discount factor $\delta_k(n)$ "connects" two regions: $V(x) \equiv V_k(x) \equiv V_{k+1}(x)$ for $0 < \delta < \delta_k(n)$, and $V(x) \equiv V_{k+1}(x) \not\equiv V_k(x)$ for $\delta_k(n) < \delta < 1$. The following theorem characterizes the knot discount factors for $n = 1$ and $n = 2$.

Theorem 2. *If $n = 1$, equation (37) has the following unique solution:*

$$\delta_k(1) = \cos^2 \frac{\pi}{k+1}, \qquad k \geq 2. \tag{38}$$

If $n = 2$, equation (37) can be simplified to

$$(1 - \delta_k(2))^2(1 - (4\delta_k(2))^k) = 1 - 4\delta_k(2), \qquad k \geq 2. \tag{39}$$

Proof. See the Appendix.

Figure 1 illustrates $\delta_k(n)$ for $n = 1$ and $n = 2$. For example, if $n = 1$ and $\delta = 0.4$, the maximum number of search periods is equal to three. This means that for any initial value of $x(0)$, in any non-trivial SMPE the project will be finished in three or fewer periods.

We can see that there is a monotonic convergence of $\delta_k \to 1$ when $k \to \infty$. This convergence is easy to prove by taking a limit $k \to \infty$, and applying it to (38) and (39). One possible interpretation of this result is that if $n = 1$ or $n = 2$ and $\delta < 1$, players search the island either for a *finite* number of periods or not at all. Note that this result is consistent with part (a) of Lemma 1.

5.5 Tragedy of the commons or free riding

We have described the efficient SMPE for the cases of $n = 1$ and $n = 2$. Let us use derived results to provide some intuition for the general problem.

Consider equation (5):

$$\Psi(x) = \max_{(n-1)I \leq y \leq x} \{(1 - x)(x - y) + \delta\Psi(z)\}.$$

Note that $I_i = (x - y)$ is the current search area for player i. We observe two main effects in the model: tragedy of the commons and free riding. The first effect works if

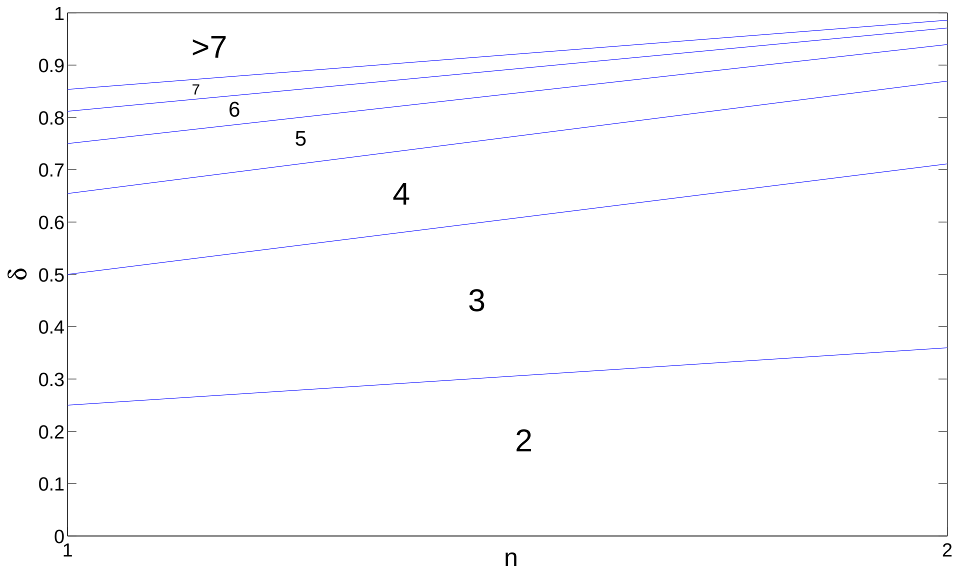

Figure 1: Different zones

the value of x is small, $0 < x < 1$. In this case the first term in equation (5) is positive, $(1-x)(x-y) = (1-x)I_i > 0$. In a multi-player case, each player has incentives to increase his current expected payoff by searching more. In the equilibrium players over-search, which is a standard tragedy of the commons effect. The second effect is present for larger values of x, namely when $x > 1$. In this case the first term in equation (5) is negative, and players have incentives to decrease their current losses and undertake insufficient search. This is a standard free-riding effect. If the island size is $x = 1$, then neither of the effects is present. Figure 2 illustrates these effects: tragedy of the commons (for $x < 1$) and free riding (for $x > 1$). For small values of x, the aggregate search, $J_n(x)$, is the same for $n = 1$ and $n = 2$; players search the whole island for just one period. On this part of the graph, both curves coincide. For larger values of x but still less than one, the curve for $n = 2$ is above the curve for $n = 1$. This means that the monopolist, $n = 1$, searches the island efficiently, while in the case of two players, $n = 2$, the tragedy of the commons effect takes place, and players search the whole island too fast. Finally, we can see that the curve for $n = 1$ is above the curve for $n = 2$ when $x > 1$. Free riding takes place on this interval, and two players search the island too slowly relative to the monopolist. Note also that due to the tragedy of the commons and the free-riding effects, there are values of x for which the monopolist finds search profitable, even though two players prefer not to search.

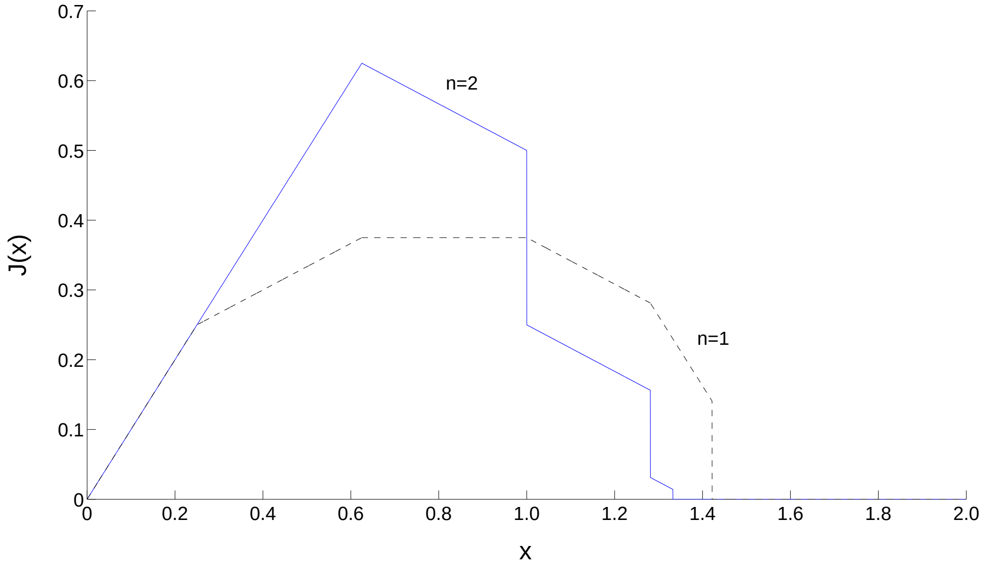

Figure 2: $J_n(x)$: $n = 1$ is a dashed line, $n = 2$ is a solid line; $\delta = 0.75$.

6 Three or More Players

If $n \geq 3$, then solving for $V_k(x)$ with the help of the value-iteration procedure is a much more tedious task.[20] Moreover, neither SMPE may be efficient for larger values of x. To illustrate the difficulties and present some interesting and unexpected results, we use the following example.

Example 1

Suppose that $n = 4$ and $\delta = 0.99$. Using the value-iteration procedure described in subsection 5.3 for $n = 1$ and $n = 2$, we derive the following quadratic polynomials[21]:

$$P_1(x) = \frac{1}{4}(-(1-x)^2 + (1-x)),$$

$$P_2(x) = \frac{1}{4}(-3.0202(1-x)^2 + 0.5(1-x) + 0.2475),$$

$$P_3(x) \approx \frac{1}{4}(-1.6689(1-x)^2 + 0.0828(1-x) + 0.2655),$$

$$P_4(x) \approx \frac{1}{4}(-2.2105(1-x)^2 + 0.0248(1-x) + 0.2639),$$

and

$$P_5(x) \approx \frac{1}{4}(-1.9139(1-x)^2 + 0.0056(1-x) + 0.2613).$$

[20]Note that $V_1(x)$ and $V_2(x)$ are derived in the previous section for any n.

[21]Note that the values for coefficients in $P_3(x)$, $P_4(x)$ and $P_5(x)$ are approximate.

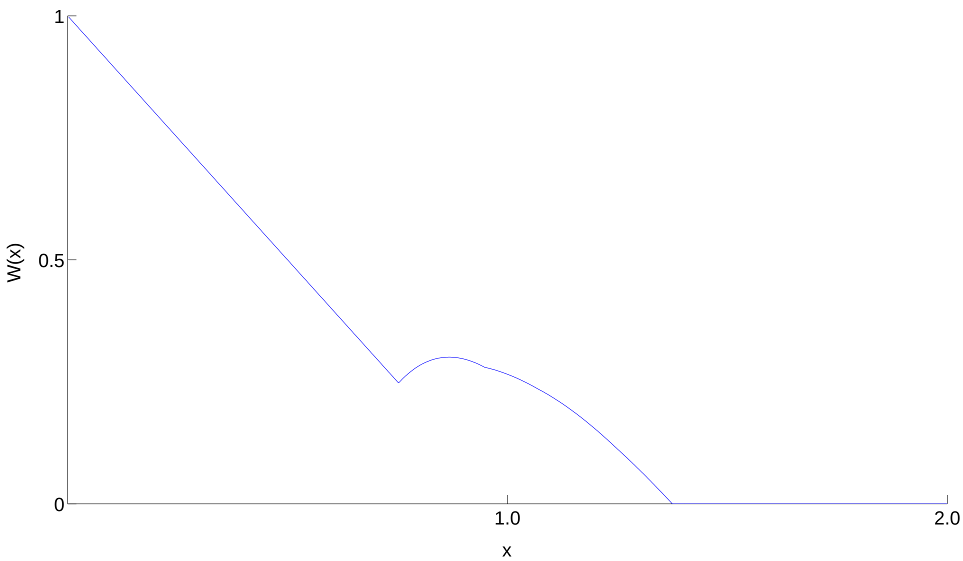

Figure 3: $W(x) = 4V(x)$.

Figure 3 illustrates the combined four-player value function ($W(x) = 4V(x)$) for $n = 4$ and $\delta = 0.99$, where the value function is given by

$$V(x) = \begin{cases} P_1(x)/x, & \text{if} \quad x \leq 0.7525, \\ P_2(x)/x, & \text{if} \quad 0.7525 < x \leq 0.9481, \\ P_3(x)/x, & \text{if} \quad 0.9481 < x \leq 1.0702, \\ P_5(x)/x, & \text{if} \quad 1.0702 \leq x \leq 1.2448, \\ P_3(x)/x, & \text{if} \quad 1.2448 < x \leq 1.3748, \\ 0, & \text{if} \quad x > 1.3748. \end{cases} \tag{40}$$

Let us consider the value function $V(x)$. First, as we can see from Figure 3, the value function is not monotonic in x. This means that a larger island could make all players better off. Second, as we can see from (40), the maximum number of search periods is five, and it is never optimal to plan searching for four periods.[22] Since the number of search periods is not monotonic in x, it is possible to finish the project more quickly when the island is larger. □

Let us call these surprising observations puzzles, and examine each in turn.

Puzzle 1. Non-monotonicity of the value function. A larger island can make all players better off.

It seems natural to expect that a larger island will make players worse off. How-

[22]We use numerical calculations to confirm that it is not optimal to search for more than five periods.

ever, consider two island sizes

$$\widetilde{x} = 0.75251 < 0.7526 = \widehat{x}.$$

Using (40), we get

$$V(0.75251) = P_2(0.75251) = 4.6563 \times 10^{-2}$$

and

$$V(0.7526) = P_2(0.7526) = 4.6586 \times 10^{-2}.$$

In this case, a larger island makes all players better off. This puzzling observation can be explained by inefficient player behavior when x is relatively small. Due to the tragedy of the commons effect, players search the smaller $\widetilde{x}$ too fast in the first period, and leave too little for the second period search. If the size of the island is increased to $\widehat{x}$, players search the island more slowly and efficiently. This efficiency improvement is large enough to outweigh the increase in the cost of searching the larger island.

The following proposition demonstrates that this result holds whenever there are at least $n = 3$ players.

Proposition 3. *For any $n \geq 3$, there exists $x > t_1$ such that $V(x) > V(t_1)$.*

Proof. See the Appendix.

Puzzle 2. Non-monotonicity of the search function. A larger island can speed up or slow down the search process.

It seems natural to expect that the maximum number of search periods monotonically increases with island size. However, consider the following island sizes

$$x_1 = 1 < x_2 = 1.1 < x_3 = 1.3.$$

Using (40), we find that searching islands of sizes x_1 and x_3 takes at most three periods while searching an island of size x_2 takes at most five periods. This implies that there is no monotonicity with respect to the number of search periods. Moreover, players never plan to search the island for at most four periods. This puzzling observation can be explained by inefficient player behavior when the island size is relatively large, and there are many players. If this effect is sufficiently strong, the efficient SMPE may not exist.

The intuition for non existence of the efficient SMPE is as follows. There are always multiple SMPE in the model but we only consider the efficient SMPE where players obtain the highest expected payoff. In such an equilibrium, players make their optimal search plans *before* the search starts. However, if the first-period search is unsuccessful, then the original search plan might no longer be optimal for the remaining unsearched area of the island. This is exactly what is happening in this example. When $x = 1.1$ players obtain the highest expected payoff if they plan to search for five periods. In the first period, they together search $J(1.1) \approx 0.0142$, which

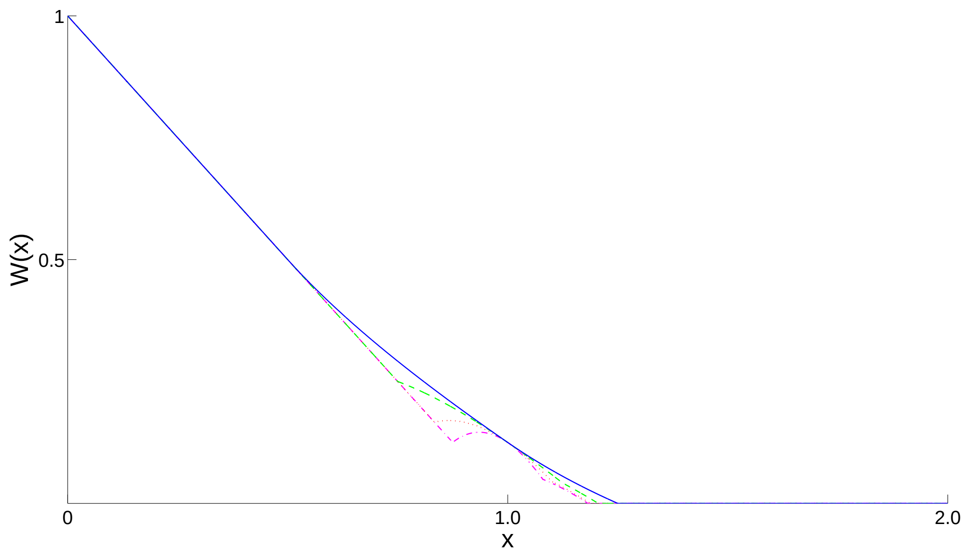

Figure 4: $W_1(x)$ is a solid line, $W_2(x)$ is a dashed line, $W_3(x)$ is a dotted line; $\delta = 0.5$.

results in new $x \approx 1.0858$. As we can see from (40), when $x \approx 1.0858$ it is optimal to search for at most five periods. This means that the original five-period search plan is based on a four-period search plan, which does not provide players with the highest expected payoff. In other words, for larger island sizes, namely $x > 1.0702$, even though some SMPE exist, none of them is efficient. Note that the value function given by (40) does not constitute SMPE because it is history dependent. Also, the non existence of the efficient SMPE is only possible with 3 or more players. This is because when there are at least 3 players, the free-riding effect may be sufficiently strong, leading to strong non monotonicity of the search function. This means that even though players always play individually optimal strategies in every state, as a team they may not. That is, equilibrium strategies may be payoff inefficient for some states.

We conclude this section with investigation of multi-player search efficiency.

Puzzle 3. Efficient search.

It is intuitive that the monopolist searches efficiently. It is also intuitive that multi-player search reproduces the monopoly outcome when the island is small; players search the whole island in just one period. However, efficient search is not limited only to small islands. If the island size is $x = 1$ and $\delta \leq 0.5$, players also search efficiently. If the combined n-player value function is defined as $W_n(x) = nV_n(x)$, then

Proposition 4. *When* $0 < \delta \leq 0.5$, $W_n(1) = W_1(1)$ *for any* $n \geq 2$.

Proof. See the Appendix.

Figure 4 illustrates Proposition 4. It shows the combined value function when one, two, three, or four players search the island and $\delta = 0.5$. Note that for $\delta \leq 0.5$ and $x \leq 1$, the project is completed in at most two periods, which means that the constructed equilibrium for $x \leq 1$ is the efficient SMPE. Note also that the total value function at $x = 1$ is the same in all four cases. Since for $x = 1$ neither the tragedy of the commons nor the free-riding effect is present, the multi-player search reproduces the monopoly outcome. This result has the following intuitive explanation. For this unique island size, players get zero expected payoff in period one of the two-period search. Consequently the objective function of players is to maximize their expected payoffs from the second period only. In the second period in the symmetric equilibrium, each firm receives a payoff proportional to the payoff of the monopoly. This guarantees that multi-player search reproduces the monopoly outcome.

7 The Unobservable Case

To contrast our findings with Bonatti and Hörner (2011) we consider the following extension where investments are unobservable. Specifically, we assume that every period each of n players searches without observing what areas have been searched by other players. To make this extension consistent with the rest of the paper, we assume that before any search is undertaken, players coordinate on how to divide the island in n equal parts. The strategy of each player is what area of his own portion of the island, and the other players' portion, to search each period. Note that if players search the areas associated with other players, there could be some duplication. In the next proposition we show that if players are sufficiently impatient, the efficient search can be supported as a Nash equilibrium. Formally,

Proposition 5. *There exists $0 < \delta^* < 1$ such that for $\forall\ \delta \leq \delta^*$, and $\forall\ n \geq 2$, $W_n(x) \equiv W_1(x)$.*

Proof. See the Appendix.

This means that not only is there less under or over-investment when player effort is unobservable (compared to the observable case), but players also search efficiently provided they are sufficiently impatient. The intuition for this result is as follows. When players are impatient, the efficient search requires covering relatively large areas each period. Given that the search choices are unobservable, it is not beneficial to search the areas associated with other players, who are assumed to search their own areas randomly; hence there is a positive probability of duplication. On the other hand, when players are sufficiently patient, the efficient search requires covering relatively small areas each period; it may be worth searching the areas of other players. Consequently, when players are sufficiently patient, they have to search faster than the monopolist.[23]

[23]The general case when players are sufficiently patient is not easily tractable. However, all our attempts to construct an example where the unobservable case leads to more inefficiency in comparison with the observable case were unsuccessful.

8 Discussion

The only dynamic R&D races that search literature has so far investigated rely on the memorylessness property of the exponential distribution. In contrast in this paper, we dealt with the case in which the success function is uniform. First, we identify two types of inefficiencies in the model: a tragedy of the commons (for small islands) and free riding (for large islands). This differs from the literature, where only one of these inefficiencies is present for a single project. Second, expanding on the previous literature, we also investigate a multi-player case with three or more players, and demonstrate that in this case there is no monotonicity; players can be better off if the race is longer, even though such a race is more costly, and a longer race may take less time to finish.

Let us discuss some of the important assumptions we made in this paper. First of all, we assume that there is an equal probability that the treasure will be located at any given point on the island. This assumption fits well the realistic case where there is no initial information on the possible location of the treasure. It also makes the analysis tractable. However, the case in which the distribution is not uniform has been discussed in the literature, see for example Fershtman and Rubinstein (1997), Choi and Gerlach (2011) and Chatterjee and Evans (2004). The expected outcome in our setting is that players start searching the parts of the island where the treasure is most likely to be found.[24] However, the possible complication here is that it is not clear when, if at all, the project will finish. For instance, the optimal strategy could be to first search areas with a relatively high chance of success and, in the event of non-discovery, suspend the search because it becomes too costly and unprofitable.

Second, we assume that the treasure can be destroyed. However, there are many situations when players can share the treasure if it is jointly discovered. If players can share the treasure, it might lead to an additional inefficiency; all players duplicate if the island is small. However, using a similar approach, one can construct the efficient SMPE for this game. We expect that it would not make the model richer except for aforementioned additional inefficiency.

Third, the solution concept in the observable case is the efficient SMPE. There also exist asymmetric Markov perfect equilibria, where in every period each player searches a different amount (they split a pie in different proportions). Using a similar approach, one can construct the efficient asymmetric Markov perfect equilibria for this game. We expect that the procedure would be more tedious but would not make the model qualitatively different. In addition, there exist other, non-Markovian symmetric equilibria. One can define "grim-trigger" strategies similar to those used in Compte and Jehiel (2004). In the case of small islands, the only symmetric equilibrium that exists is the efficient SMPE. Players can always finish the search in one period, so "grim-trigger" strategies are not useful in supporting cooperation. In contrast in the case of large islands, with a large number of potential investment periods, "grim-trigger" strategies can improve cooperation. However, given that the search is

[24] Choi and Gerlach (2011) show that players start from the easiest project in R&D competition.

inefficient for small islands, it will be inefficient for large islands as well. Consequently, in the observable case efficient search can not be supported even with "grim-trigger" strategies.

Fourth, the assumption about one prize is very common in the theoretical R&D literature.[25] We make an additional assumption that the prize has to be located on the island. However, given that players are assumed to be risk-neutral, one can always qualify this assumption by assuming that the prize is located on the island with some positive probability, and the players are searching for the expected value of the prize.

Finally, there are some situations when a player benefits even if another player finds the treasure. For example, by imitation a rival may be able to capture some of the profits generated by an innovation. It could be an interesting extension of our project when players have positive or negative externalities on each other. We leave this extension for future research.

Appendix

Proof of Lemma 1

First, if $n = 1$, part (b) of the Lemma follows directly from the contraction mapping theorem (see for example Stokey, Lucas and Prescott (1989)). However this theorem is not useful for part (a) or when a multi-player game is considered. Consequently, we will provide a different proof, which will cover both single-player and multi-player games. For future reference note that given that the function $(1-x)(x-y)$ in equation (5) is continuous, $\Psi(x)$ is also continuous.[26]

To prove (a), first consider the case when x is arbitrarily small and apply an argument similar to Admati and Perry (1991). Specifically, from equation (5) the marginal benefit of searching additional area in the first period is equal to the value that is arbitrarily close to 1, while the marginal benefit of postponing the search by one period is less than $\delta < 1$. Consequently, it is a dominant strategy for each player to finish the search in one period.

For the remaining values of x, suppose (a) is not satisfied. First, if $I(t) = 0$ for some $t = \hat{t}$, stationarity implies investment remains zero for $\forall\ t > \hat{t}$. Hence, there are finitely many search periods. Assuming that the game lasts an infinite number of periods, this means that investment in some periods will take arbitrarily small values, i.e. for $\forall\ \varepsilon_0 > 0$ $\exists\ \varepsilon \in (0, \varepsilon_0)$ and $\exists\ t \in N$ such that $I(t) = \varepsilon$ and $n\varepsilon \ll x$. From (5) it follows:

$$\Psi(x) = (1-x)\varepsilon + \delta\Psi(x - n\varepsilon). \tag{41}$$

Expanding equation (41) in Taylor series around $\varepsilon = 0$ implies

$$\Psi(x) = -\frac{(1-x)\varepsilon}{1-\delta} + O(\varepsilon) = O(\varepsilon). \tag{42}$$

[25] See Reinganum (1989) for more detailed discussion about this assumption.

[26] See for example proposition 1 in Ericson and Pakes (1995) for a standard technique to prove a similar result.

This means that $\Psi(x)$ could be arbitrarily close to zero. For values of $x < 1$ which are not arbitrarily small this is not possible, because each player will have incentive to deviate by choosing $I = x-(n-1)\varepsilon$ and generating $\Psi(x) = (1-x)(x-(n-1)\varepsilon) > 0$. When $x = 1$ a similar argument applies; in the first period a player has an incentive to deviate by making a positive contribution, for example $I = 0.5$. In this case, $\Psi(1) = \delta\Psi(0.5-(n-1)\varepsilon)$. Given that $\Psi(x)$ is not arbitrarily close to zero when $x < 1$, $\Psi(1)$ is also not arbitrarily close to zero.

When $x > 1$ a slightly different proof is used. For a given size of island $x(0) > 1$, each player makes a positive search $I > 0$ in the first period. Assuming that the game lasts an infinite number of periods, the investments have to take arbitrarily small values, i.e. for $\forall\ \varepsilon_0 > 0\ \exists\ \varepsilon \in (0, \varepsilon_0)$ and $\exists\ t \in N$ such that $I(t) = \varepsilon$, where $x > 1$. From equation (42) it follows that when $I(t) = \varepsilon$, $\Psi(x)$ is arbitrarily small. However, in the first period players make a positive search $I > 0$, which means they incur an immediate negative payoff of $(1 - x(0))I$. Subsequently, they make additional searches which add additional negative payoffs. Overall, this argument implies that $\Psi(x(0))$ is negative, which is a contradiction.

Thus, $I(t)$ cannot be arbitrarily close to zero, which means there exists $\zeta > 0$ such that $I(t) > \zeta\ \forall\ t$. Consequently in any non-trivial SMPE the project has to be finished in a finite number of periods. This establishes part (a) of the Lemma.

Given that the project has to be finished in a finite number of periods, backward induction can be applied. Namely, we assume that $\Psi_0 \equiv 0$ and derive $\Psi_1 \equiv B\Psi_0$. This allows us to find all potential SMPE of the game if players could search for at most one period. Then we derive $\Psi_2 \equiv B\Psi_1$, which allows us to find all potential SMPE of the game if players could search for at most two periods. We continue this process until T is reached and find all potential SMPE of the game if players could search for at most T periods. Note that if $n = 1$ there always exists a unique equilibrium. If $n > 1$ SMPE is not necessarily unique. This establishes part (b) of the Lemma and concludes the proof. □

Proof of Proposition 1

Derivation of a_k, b_k and c_k

Let us show that when $n = 1$, equation (33) is the solution to the system of difference equations (31).

Define

$$R_k = v^k \cdot \prod_{j=1}^{k} a_j \qquad k = 1, 2, \ldots \tag{43}$$

Using (31), one gets the following second-order difference equation:

$$R_{k+1} = vR_k \cdot \left(-1 - \frac{1}{4\delta a_k}\right) = -vR_k - \frac{1}{4}R_{k-1} \quad k \geq 2. \tag{44}$$

The initial conditions are $R_0 = 1$ and $R_1 = -v$. The characteristic equation $4z^2 + 4vz + 1 = 0$ has two complex roots

$$z_1 = \frac{-v + ir}{2}, \quad z_2 = \frac{-v - ir}{2}, \qquad r = \sqrt{1 - v^2} > 0.$$

Denote $\varphi = \{\arg z_1 \in [0, \pi/2]\} = \arccos v$; then $z_{1,2} = -\frac{e^{\pm i\varphi}}{2}$. Further, write the solutions to equation (44) in the form $R_k = Az_1^{k+1} - Bz_2^{k+1}$, and use the initial conditions to get $A = B = -\frac{i}{\sin\varphi}$. Consequently

$$R_k = -\frac{i}{(-2)^{k+1}\sin\varphi}(e^{i(k+1)\varphi} - e^{-i(k+1)\varphi}) = -\frac{\sin[(k+1)\varphi]}{(-2)^k \sin\varphi}. \tag{45}$$

Apply (43) and (31) to get

$$a_k = \frac{R_k}{vR_{k-1}} = -\frac{\sin(k+1)\varphi}{2v\sin k\varphi}, \tag{46}$$

$$b_k = -\frac{b_{k-1}}{2a_{k-1}} = \frac{v^{k-1}\sin\varphi}{\sin k\varphi}, \tag{47}$$

$$c_k = \delta\left[c_{k-1} - \frac{b_{k-1}^2}{4a_{k-1}}\right] = \frac{v^{2k-1}\sin(k-1)\varphi}{2\sin k\varphi}. \tag{48}$$

Derivation of t_k

To find t_k, one needs to solve the quadratic equation $P_k(t_k) = P_{k+1}(t_k)$; namely

$$(a_{k+1} - a_k)(1-t_k)^2 + (b_{k+1} - b_k)(1-t_k) + c_{k+1} - c_k = 0, \quad k \geq 1. \tag{49}$$

Substitute a_k from (46) to derive

$$a_{k+1} - a_k = \frac{\sin(k+1)\varphi}{2v\sin k\varphi} - \frac{\sin(k+2)\varphi}{2v\sin(k+1)\varphi} = \frac{\sin^2(k+1)\varphi - \sin k\varphi \sin(k+2)\varphi}{2v\sin k\varphi\sin(k+1)\varphi} = \frac{\sin^2\varphi}{2v\sin k\varphi\sin(k+1)\varphi}.$$

Substitute b_k from (47), and note that $v = \cos\varphi$ to derive

$$b_{k+1} - b_k = \frac{v^k\sin\varphi}{\sin(k+1)\varphi} - \frac{v^{k-1}\sin\varphi}{\sin k\varphi} = \frac{2v^k\sin\varphi(\cos\varphi\sin k\varphi - \sin(k+1)\varphi)}{2v\sin k\varphi\sin(k+1)\varphi} = \frac{-2v^k\sin^2\varphi\cos k\varphi}{2v\sin k\varphi\sin(k+1)\varphi}.$$

Substitute c_k from (48), and note that $v = \cos\varphi$ to derive

$$c_{k+1} - c_k = \frac{v^{2k+1}\sin k\varphi}{2\sin(k+1)\varphi} - \frac{v^{2k-1}\sin(k-1)\varphi}{2\sin k\varphi} = \frac{v^{2k}(\cos^2\varphi\sin^2 k\varphi - \sin(k+1)\varphi\sin(k-1)\varphi)}{2v\sin k\varphi\sin(k+1)\varphi} = \frac{v^{2k}\sin^2\varphi\cos^2 k\varphi}{2v\sin k\varphi\sin(k+1)\varphi}.$$

Substitute the above relationships into (49), and cancel the non-zero common term $\frac{\sin^2\varphi}{2v\sin k\varphi\sin(k+1)\varphi}$ to derive

$$(1-t_k)^2 - 2v^k\cos k\varphi(1-t_k) + v^{2k}\cos^2 k\varphi = (1 - t_k - v^k\cos k\varphi)^2 = 0.$$

Consequently,

$$t_k = 1 - v^k\cos k\varphi.$$

Note that both solutions to $P_k(t_k) = P_{k+1}(t_k)$ coincide, which means that the solution is unique.

Derivation of u_k

To find u_k, one needs to solve the quadratic equation $P_k(u_k) = 0$, namely

$$a_k(1-u_k)^2 + b_k(1-u_k) + c_k = 0, \quad k \geq 1. \tag{50}$$

Substitute (46), (47) and (48) into (50) to get

$$-(1-u_k)^2 \sin{(k+1)\varphi} + 2(1-u_k)v^k \sin\varphi + v^{2k} \sin{(k-1)\varphi}.$$

Solving this quadratic equation results in

$$u_k = 1 + \frac{v^k(\sin k\varphi - \sin\varphi)}{\sin{(k+1)\varphi}}.$$

This concludes the proof. □

Proof of Proposition 2

Derivation of a_k, b_k and c_k

Let us show that when $n = 2$, formula (35) describes the solution to the system of difference equations (31). It is straightforward to derive $a_k = -1$ and $b_k = \frac{1}{2^{k-1}}$. The expression for c_k in (31) can be simplified to

$$c_k = \delta(c_{k-1} + 1/4^{k-1}). \tag{51}$$

Introduce a new variable $e_k = c_k 4^k$. Equation (51) transforms to

$$e_k = 4\delta(e_{k-1} + 1),$$

where $e_1 = 0$. The solution to this linear difference equation is $e_k = \frac{4\delta-(4\delta)^k}{1-4\delta}$. Substitute $c_k = e_k/4^k$ to derive

$$c_k = \frac{4\delta - (4\delta)^k}{(1-4\delta)4^k}.$$

Derivation of t_k

To find t_k, one needs to solve the quadratic equation $P_k(t_k) = P_{k+1}(t_k)$, namely

$$a_k(1-t_k)^2 + b_k(1-t_k) + c_k = a_{k+1}(1-t_k)^2 + b_{k+1}(1-t_k) + c_{k+1}, \quad k \geq 1.$$

From equation (35), $a_k = a_{k+1} = -1$; consequently,

$$t_k = 1 + \frac{c_{k+1} - c_k}{b_{k+1} - b_k}.$$

Substitute b_k and c_k from equation (35) to derive the following indifference points

$$t_k = 1 - \frac{3\delta + (4\delta)^k(\delta - 1)}{2^k(4\delta - 1)}.$$

Note that the solution to the quadratic equation $P_k(t_k) = P_{k+1}(t_k)$ is always unique.

Derivation of u_k

To find u_k, one needs to solve the quadratic equation $P_k(u_k) = 0$, namely

$$a_k(1-u_k)^2 + b_k(1-u_k) + c_k = 0, \quad k \geq 1.$$

Substituting $a_k = -1$ from equation (35) and solving the above quadratic equation gives

$$u_k = 1 + \frac{\sqrt{b_k^2 + 4c_k} - b_k}{2}. \tag{52}$$

Note that with the help of (35), one can simplify:

$$b_k^2 + 4c_k = \frac{(4\delta)^k - 1}{4^{k-1}(4\delta - 1)}. \tag{53}$$

Substitute equation (53) into equation (52) to get

$$u_k = 1 + \frac{\sqrt{\frac{1-(4\delta)^k}{1-4\delta}} - 1}{2^k}. \tag{54}$$

This concludes the proof. □

Proof of Lemma 2

First, let us show that condition (24) is satisfied; that is, $t_{k-2} \leq z \leq u_{k-1}$. Let us prove the first part $z \leq u_{k-1}$ by contradiction, assuming that $z > u_{k-1}$. Refer to equation (22), which is written below:

$$\Psi_k(x) = \max_{(n-1)I \leq y \leq x} \{(1-x)(x-y) + \delta\Psi_{k-1}(z)\}.$$

Given $x \geq y \geq z > u_{k-1} \geq \ldots \geq u_2 \geq u_1 = 1$, it follows that the first term $(1-x)(x-y)$ has to be non-positive. If $z > u_{k-1}$, then the second term $\delta\Psi_{k-1}(z)$ is negative. That means the whole expression on the right of equation (22) has to be negative. Obviously that could not be an optimal choice for a player because by choosing $y = x$, that is, by not searching, a player can get the value of zero. Consequently, there is a contradiction, and condition $z \leq u_{k-1}$ is proved.

Now let us show that $t_{k-2} \leq z$. When $n = 1$, we prove this inequality by contradiction, assuming that $z < t_{k-2}$. Note that by construction, when $z < t_{k-2}$, the following condition holds: $P_{k-1}(z) < P_{k-2}(z)$. That implies that instead of using the original k-period path (searching $x-y$ in the first period and making a further $k-1$ searches according to $P_{k-1}(z)$), the monopolist could use a $(k-1)$-period path (searching the same amount $x-y$ in the first period, and making a further $k-2$ searches according to $P_{k-2}(z)$), and increase the value. Refer to equation (22): both paths have the same first term, while the second term is larger for the $(k-1)$-period path. This implies that the k-period path does not improve the value in comparison with the optimal $(k-1)$-period path, which means that whenever $z < t_{k-2}$, the k-period path is not optimal. Condition $t_{k-2} \leq z$ is thus proved when $n = 1$.

When $n = 2$, let us prove the above condition directly. From equation (27) and Proposition 2, it follows that $z(x) = 1 - 2^{1-k} - \frac{1-x}{\delta}$. It is easy to see that $z(x)$ is a monotonically increasing function in x. Consequently, it is sufficient to prove the above condition for $x = t_{k-1}$. Substituting values derived in Proposition 2 and simplifying gives

$$\frac{3\delta + (4\delta)^{k-1}(\delta - 1)}{\delta(4\delta - 1)} \leq \frac{1 + 2\delta + 2(4\delta)^{k-2}(\delta - 1)}{4\delta - 1}.$$

Further simplifications result in $\frac{(4\delta)^{k-2}-1}{4\delta-1} \geq 0$, which is satisfied for any $k > 2$.

Second, let us show that condition (25) is satisfied; that is, $\Psi''_{k-1}(z) < 0$. From equation (30), it is clear that the sufficient condition for $\Psi''_{k-1}(z) < 0$ is that $P''_{i-1}(z) < 0 \ \forall\ i = 2, \ldots, k-1$. From equation (26), it is easy to see that the above condition is equivalent to $a_{i-1} < 0 \ \forall\ i = 2, \ldots, k-1$. For $n \geq 2$, it is straightforward to show that $s \geq 0$. From (31), one can see that a_k is a sum of two negative numbers; consequently, it has to be negative.

Now let us prove this condition for $n = 1$. Substitute t_k from (34) into (26) to get

$$P_k(t_k) = \frac{v^{2k-1}}{2n \sin k\varphi}(-\sin{(k+1)\varphi} \cos^2 k\varphi + 2 \sin\varphi \cos k\varphi + \sin{(k-1)\varphi}). \tag{55}$$

Use the fact that

$$\sin{(k+1)\varphi} = \sin{(k-1)\varphi} + 2\sin\varphi\cos k\varphi$$

and substitute into (55) to derive

$$P_k(t_k) = \frac{-\delta^k a_k \sin^2 k\varphi}{n}. \tag{56}$$

Wherever the value function at t_{k-1} is positive, a_{k-1} has to be negative.

Next, for the proof that condition (29) has a unique solution, see the proofs of Propositions 1 and 2. In the case of $n = 1$, both solutions to (29) coincide; while in the case of $n = 2$, the second solution does not exist.

Finally, given that all necessary and sufficient conditions are satisfied, results derived in Theorem 2 and Propositions 1 and 2 are consistent. In particular, this means that if $a_{k-1} = 0$, the optimal path includes $k-1$ steps only. A direct way of proving this result is to notice that if $a_{k-1} = 0$ in equation (56), it results in $P_{k-1}(t_{k-1}) = 0$, which means that the k-th step is unnecessary. □

Proof of Theorem 2

First, let us prove the result for $n = 1$. Equation (37) (which defines δ_k) is equivalent to the condition $R_k = 0$ (R_k is defined in (43)). Apply (45) to get

$$\varphi_k \quad : \qquad (k+1)\varphi = n\pi, \tag{57}$$

where $n \geq 1$ is some integer which can be different for different values of k, i.e. $n = n_k$. Let us prove by induction that $n_k = 1 \ \forall\ k$. It is easy to see that for $k = 2$ the statement is correct, i.e. $\varphi_2 = \pi/3$ and $\delta_2(1) = \cos^2 \pi/3 = 1/4$. Substitute $k' = k+1$ in (57) to get

$$n_{k+1}\pi = (k+1)\varphi_{k+1} \leq (k+1)\varphi_k = \frac{k+1}{k}\pi < 2\pi.$$

Note that we are using the inductive hypothesis that $n_k = 1$ and the fact that $\varphi_k = \arccos\sqrt{\delta_k}$ is monotonically decreasing in k. Given that n_{k+1} is an integer, it must be the case that $n_{k+1} = 1$. Substitute $n = 1$ in (57) to get $\varphi_k = \pi/(k+1)$, which means $\delta_k(1) = \cos^2 \frac{\pi}{k+1}$, $k \geq 2$.

Next let us prove the result for $n = 2$. Substitute t_k from equation (36) and u_k from equation (54) into equation (37) to get

$$1 + \frac{\sqrt{\frac{1-(4\delta)^k}{1-4\delta}} - 1}{2^k} = 1 - \frac{3\delta + (4\delta)^k(\delta - 1)}{2^k(4\delta - 1)}.$$

Simplify the above expression to

$$\sqrt{\frac{1-(4\delta)^k}{1-4\delta}} - 1 = \frac{3\delta + (4\delta)^k(\delta - 1)}{(4\delta - 1)}.$$

Further simplifications give

$$\sqrt{\frac{(4\delta)^k - 1}{4\delta - 1}} = \frac{((4\delta)^k - 1)(1 - \delta)}{(4\delta - 1)},$$

and

$$\sqrt{\frac{(4\delta)^k - 1}{4\delta - 1}}(1 - \delta) = 1. \tag{58}$$

Square both sides of equation (58) to derive equation (39). This concludes the proof. □

Proof of Proposition 3

When $x = t_1$ players are indifferent between searching the island for two periods or for one period:

$$V_1(t_1) = V_2(t_1).$$

Let us show that $V_2'(t_1) > 0$ for $n \geq 3$, which means that there exists a value of x which is "slightly" larger than t_1 ($x > t_1$) such that $V(x) = V_2(x) > V(t_1)$:

$$V_2'(t_1) = \left(\frac{P_2(x)}{x}\right)'_{x=t_1} > 0. \tag{59}$$

Applying (16) and (19) transforms inequality (59) to

$$V_2'(t_1) = a_2 - \frac{a_2 + b_2 + c_2}{(1 - \delta/n)^2} > 0. \tag{60}$$

With the help of (16) and (17), inequality (60) simplifies to

$$n^2(2n - 6 - 2\delta) + \delta(10n - 4\delta) > 0. \tag{61}$$

When $n \geq 4$, the above inequality always holds because both terms on the left are positive. When $n = 3$, inequality (61) simplifies to

$$\delta(12 - 4\delta) > 0,$$

which is also valid. The proposition therefore is proved. □

Proof of Proposition 4

Let us prove this proposition in two steps. First, let us show that when $0 < \delta \leq 0.5$, knot $t_2 \geq 1$ for any $n \geq 1$, which implies $V(1) = V_2(1)$ for any n. Applying (34) when $n = 1$ gives $t_2 = 1 - v^2 \cos 2\varphi = 1 - \delta(2\cos^2\varphi - 1) = 1 + \delta(1 - 2\delta)$. It is easy to see that $t_2 \geq 1$ when $0 < \delta \leq 0.5$. Applying (36) when $n = 2$ gives $t_2 = 1 - \frac{3\delta + 16\delta^2(\delta - 1)}{4(4\delta - 1)} = 1 + \frac{3}{4}\delta - \delta^2$. It is easy to see that $t_2 \geq 1$ when $0 < \delta \leq \frac{3}{4}$. Applying (31) when $n \geq 3$, one can derive $a_2 = -1 - s$, $b_2 = \frac{1}{2}$, $c_2 = \frac{\delta}{4}$, $a_3 = -\frac{1+2s}{1+s}$, $b_3 = \frac{1}{4(1+s)}$ and $c_3 = \frac{\delta^2}{4} + \frac{\delta}{16(1+s)}$. When $n \geq 3$, it is easy to see that $a_3 - a_2 = 1 + s - \frac{1+2s}{1+s} = \frac{s^2}{1+s} > 0$, $b_3 - b_2 = \frac{1}{4(1+s)} - \frac{1}{2} < 0$, and finally $c_3 - c_2 = \frac{\delta}{16(1+s)} - \frac{\delta}{4}(1 - \delta) = \frac{\delta(4\delta^2 - 3\delta - (1-\delta)n(n-2))}{4(4\delta + n(n-2))} < 0$ when $0 < \delta \leq \frac{3}{4}$. To find t_2, one needs to solve the quadratic equation $P_2(t_2) = P_3(t_2)$, namely

$$a_2(1 - t_2)^2 + b_2(1 - t_2) + c_2 = a_3(1 - t_2)^2 + b_3(1 - t_2) + c_3,$$

which gives

$$t_2 = 1 + \frac{b_3 - b_2 + \sqrt{(b_3 - b_2)^2 - 4(a_3 - a_2)(c_3 - c_2)}}{2(a_3 - a_2)}.$$

It is easy to see that $t_2 > 1$ when $0 < \delta \leq \frac{3}{4}$.

Second, one needs to show that $nV_2(1)$ is the same for any n. Applying (26) gives $nV_2(1) = c_2 = \frac{\delta}{4}$. The proposition is thus proved. □

Proof of Proposition 5

Let us prove the statement for $\delta^* = \frac{1}{4}$. In this case the efficient equilibrium involves at most two periods, see equation (38). First, given that in the proposed equilibrium both players search efficiently, they have no incentive to deviate by changing the division of search between the first and the second period – this is the optimal search strategy they could adopt. Second, let us show that they also have no incentive to search the areas associated with the other players. They can deviate from the proposed equilibrium, by searching these areas either in the first or in the second period. Given that in the second period the other players finish the search in their areas, searching these areas in the second period is clearly disadvantageous. On the other hand, searching these areas in the first period by ϵ would generate additional payoff of

$$-\epsilon + \frac{\epsilon(x - I)}{x^2}, \tag{62}$$

where I is given in (21) when $n = 1$. The first term is the cost of searching the other players' areas, while the second term is the expected benefit to the deviator who searches ϵ, while the 'incumbent' searches I. Given that the search by the 'incumbent' is unobservable, it is assumed that from the deviator's prospective, the 'incumbent' searches his area randomly. One can show that the combined effect given by (62) is negative for any positive ϵ when $\delta \leq \frac{1}{4}$. Consequently neither player has an incentive to deviate by searching the other players' areas. □

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