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Treasure Game

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Treasure game*

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Abstract

A prize is located at an unknown point on an island. In every period, each of n players searches part of the previously unsearched portion of the island. If a player finds the prize alone he wins it and the game ends. Players have a per-period discount factor and a search cost proportional to area searched. Superior symmetric Markov perfect equilibria are characterized. Equilibria for $n \geq 2$ can exhibit two types of inefficiency: a tragedy of the commons (for small islands) and free riding (for large islands). For $n \geq 3$, equilibrium properties are non-monotonic: players may be better off searching larger islands. The model is very general: applications include R&D races, team production, and extraction of exhaustible resources.

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1 Introduction

Consider pharmaceutical firms participating in an R&D race for drug discovery, paparazzi looking for a movie star in city hotels, or researchers looking for solutions to the six Millennium Prize Problems in mathematics. All these situations are examples of a general problem, namely, a treasure hunt where the prize value is common knowledge, but the search costs are unknown ex ante. This is the problem we consider in this paper.

We analyze a dynamic model in which a given number of players search for a treasure hidden somewhere on an island of a given area. The value of the treasure is

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common knowledge, and search is costly. Once the treasure is found the game ends. In each period, all players make their search decisions simultaneously. If the treasure is not found in the current period, search in the next period occurs over the remaining unsearched area. If several players find the treasure simultaneously (search the same part of the island), each of them incurs their costs, but the treasure will be destroyed (players do not get any treasure).[1] We consider the case when search is observable; that is, during the search players are informed about the areas that have already been searched by their opponents.[2]

We analyze a game in which each state is described by the remaining unsearched area. While there are multiple subgame perfect equilibria in the game, we restrict our attention to the symmetric Markov perfect equilibria (SMPE).[3] Among all SMPE, we only consider the Superior SMPE (SSMPE); that is, the SMPE with the highest total expected payoff in the absence of collusion. We find and completely characterize the unique SSMPE when there are one or two players. For more than two players we characterize the unique SSMPE when the maximum number of search periods is two.

We compare the SSMPE for multi-player search to the case of monopoly. Relative to the latter, multi-player search is typically inefficient except for very small islands when players behave as a cartel and search lasts just one period. In general, there are two types of potential inefficiency. First, in the case of small islands, multiple players search too fast; the probability of finding a treasure is relatively high, which means players have an incentive to over-search in the current period. This is similar to a standard *tragedy of the commons* effect. It leads to over-investment in comparison with the case of monopoly. Second, in the case of large islands, players undertake insufficient search; the probability of finding the treasure is relatively low, so the immediate payoff from search is negative. Players want others to search and incur current losses, hoping that the treasure will not be found in the current period. In other words, there is an incentive to postpone search to a future period, when it will be more profitable. This is similar to a standard *free-riding* effect. It leads to under-investment in comparison with the monopolist. Note that in the present model, in contrast to the existing literature, both the tragedy of the commons and the free-riding effect may endogenously arise within the same project.

Since search is costly, it seems natural to conjecture that a smaller island (lower search costs) is better than a bigger island for all players. In fact, it turns out not to be the case. By example, we illustrate that players might be worse off with a

[1]This assumption is standard in the R&D literature. Intuitively, if several players discover the treasure simultaneously, fierce competition between them runs down the surplus to zero. A good example of such a situation for just two players is Lockheed and Douglas jet development in the 1960s. For more detail, see *The Economist (1985)*; and Chatterjee and Evans (2004). Many examples of simultaneous discoveries in science can be found in Merton (1973). See Section 5 for an alternative assumption.

[2]We relax this assumption in Section 5.

[3]Imposing Markov perfection not only makes our analysis simpler, while still being consistent with rationality, but it also makes our results directly comparable to those in the previous literature. See Maskin and Tirole (1988), Bhaskar et al. (2010) and Battaglini et al. (2012) for a general discussion of why the use of SMPE is appropriate.

smaller island. This surprising observation means that an increase in expected costs might make all the players better off. It has the following intuitive explanation. If the island is small, the tragedy of the commons effect is strong, and players oversearch the island. If the island area is increased, the tragedy of the commons effect decreases, and players search the island more efficiently. It turns out that with three or more players this efficiency improvement may be large enough to outweigh the increase in the cost of searching the larger island.

In the special case when the cost of searching the entire unsearched area of the island is exactly equal to the treasure value, the tragedy of the commons and free-riding effects could be absent, in which case multi-player search reproduces the monopoly outcome. This happens when the discount factor is sufficiently low, guaranteeing that players search the island in at most two periods. For this unique island area, players get zero expected payoff in period one. Consequently the objective function of players is to maximize their expected payoffs from the second period only. In the second period in the symmetric equilibrium, each firm effectively receives an equal share of the monopoly payoff. This guarantees that multi-player search reproduces the monopoly outcome.

There are two alternative interpretations of our model. First, as a dynamic game of financing a private good with an uncertain threshold. Barbieri and Malueg (2010) introduce threshold uncertainty into a private-values one-period model of voluntary provision of a discrete public good. In contrast to their model, we consider a dynamic game where a private good rather than a public good is being financed. Second, the reduced form generated in our model resembles the problem of extraction of exhaustable resources under common access or, equivalently, a multi-player cake-eating problem.[4]

In contrast to much of the existing literature, we focus on the dynamics of investment in relation to private goods. Previous work typically investigates dynamic models of investment in relation to contributions to public goods.[5] Georgiadis (2014) is the work that is closest in spirit to our paper. In his framework, agents contribute to a public project that gradually progresses towards completion. The main finding is that members of a larger team work harder than those of a smaller team if and only if the project is sufficiently far from completion. The public-good nature of the problem implies that there always exists free-riding and agents always under-invest. In contrast, in our private-good framework both under-investment and over-investment are possible.

Previous work on private goods mostly deals with situations that are either static or involve complete information.[6] The typical outcome of these models is that firms

[4]See for example Long (2011) for a recent survey of dynamics games in the economics of natural resources.

[5]See, for example, Admati and Perry (1991), Marx and Matthews (2000), Lockwood and Thomas (2002), Compte and Jehiel (2004), Bonatti and Hörner (2011), Battaglini et al. (2012) and Matthews (2013).

[6]See, for example, Loury (1979), Dasgupta and Stiglitz (1980a, b) and Lee and Wilde (1980), and Long (2010) for a literature survey.

overinvest. One of the few papers that considers the dynamics of investment is Reinganum (1981), who shows that in a dynamic R&D race where each firm chooses a time path of expenditures, firms may underinvest as compared to the monopoly outcome. Aggregate expenditure on R&D may, therefore, depending on the exogenous parameters, be either too high or too low relative to the monopoly outcome. To simplify the analysis, Reinganum assumes that the success function is exponential. As a result, previously acquired knowledge does not change the probability of current success in the race; that is, the equilibrium strategies may be time-independent. There are many situations, however, where the memorylessness assumption is not satisfactory; for example, when the search domain, while potentially large, is finite. Consideration of this case permits us to shed light on the dynamics of when over-investment or under-investment are likely to arise.[7]

Chatterjee and Evans (2004) analyze an R&D race, where two competing firms simultaneously choose between two research projects, where investment is observable and it is common knowledge that exactly one of these projects will be successful if enough investment is made. While agents in their model decide which area to search (how much they search each period is exogenously determined), agents in our model decide how much to search (the location has no importance).

Our paper is related to the literature on strategic experimentation with publicly observable actions and outcomes.[8] This strand of literature uses the two-armed bandit framework to model the trade-off between experimentation and exploitation in teams. In particular, Klein and Rady (2011) assume a negative correlation of the quality of the risky arm across players. Note that strategic interaction in their model arises out of purely informational concerns. In our context, Klein and Rady assume that a player benefits from the other player's previous periods of unsuccessful search. However, contrary to us, there is no payoff rivalry among players. In their model all Markov perfect equilibria are in cutoff strategies. In contrast, in our framework once players begin searching they stop only if they find the treasure.

Finally, there is a literature that models research contests as rank-order tournaments. In contrast to our paper, this literature analyzes the situation in which there are multiple potential innovations that compete against each other. Some examples recently discussed in the literature include: a 1992 refrigerator competition (see Taylor, 1995), a 1829 steam locomotion tournament (see Fullerton and McAfee, 1999), a 1714 British contest for a method of determining longitude at sea (see Che and Gale, 2003). As with the literature on strategic experimentation, and in contrast to our model, all Markov perfect equilibria in these papers are in cutoff strategies.

The paper is organized as follows. A simple example is presented in Section 2. Section 3 introduces the model. A general procedure for finding the SSMPE is derived in Section 4. Section 5 presents some extensions, while Section 6 discusses the results

[7]Doraszelski (2003) investigates the parallel question of when the firm that is behind in the race engages in catch-up behavior. Using simulation, he obtains richer investment dynamics by discarding the memorylessness assumption. In this paper we derive analytical results in the same vein.

[8]See, for example, Bolton and Harris (1999), Keller, Rady and Cripps (2005), Keller and Rady (2010) and Klein and Rady (2011).

and their robustness to alternative assumptions.

2 An Example

Consider the following two-period game. Player $i = 1, 2$ may search the island of size $x_1 = 1$ in two periods, $t = 1, 2$, in order to find a treasure of value 2. Denote the area player i searches in period t as I_t^i. The cost of searching for either player is linear, $C(x) = x$, and there is an equal probability that the treasure will be located at any given point on the island. If players search the same area and find the treasure simultaneously, each of them incurs their costs, but the treasure will be destroyed. Each player observes how much the other player previously searched, before making his search plans. Payoffs from the second period are discounted at a common factor $\delta = 0.75$.

First, consider the case when only one player (a monopolist) searches the island in two periods. Let us solve the game by backward induction. Consider the second-period subgame where the treasure is not found in the first period and the unsearched area of the island is $x \leq x_1$. Note that if the monopolist searches I_2 in the second period, then his expected second-period payoff is

$$2I_2/x - I_2 = I_2(2/x - 1).$$

Since $x \leq x_1 = 1$ the monopolist has a dominant strategy to search as much as possible in the second period. Therefore, $I_2 = x$ and the monopolist's value in the second-period subgame with an unsearched area of x is

$$V_1(x) = 2 - x.$$

Consider the first period now. If the monopolist searches I_1 in the first period, his expected payoff from the two-period game is

$$2I_1/x_1 - I_1 + \delta(1 - I_1/x_1)V_1(x_1 - I_1),$$

where the monopolist incurs searching cost of I_1, gets immediate expected payoff of $2I_1/x_1$, and anticipates the expected payoff of $\delta(1 - I_1/x_1)V_1(x_1 - I_1)$ in the second period. Simplifying the above expression yields

$$I_1 + \delta(1 - I_1)(1 + I_1) = I_1 + \delta(1 - (I_1)^2).$$

The first-order condition with respect to I_1 gives

$$1 - 2\delta I_1 = 0.$$

Consequently, there is a unique SPE that involves searching 2/3 of the island in the first period and the remaining 1/3 of the island in the second period if necessary; that is, $I_1 = 2/3$ and $I_2 = 1/3$. The value for the monopolist of the two-period game with the island size of $x_1 = 1$ is $V_2(x_1) = \frac{13}{12}$.

Next, consider the case when two players search the island in two periods. Let us solve the game by backward induction. Consider the second period subgame where the treasure is not found in the first period and the unsearched area of the island is $x \leq x_1$. Note that if player i searches I_2^i in the second period and there is no duplication, then his expected second period payoff is

$$2I_2^i/x - I_2^i = I_2^i(2/x - 1).$$

Provided there is no duplication, each player has a dominant strategy to search as much as possible in the second period. Therefore in the unique symmetric SPE, $I_2^1 = I_2^2 = x/2$ and the value for player i in the subgame with the unsearched area of x is

$$\hat{V}_1(x) = 1 - x/2.$$

Now consider the first period. Note that if player i searches I_1^i in the first period, then his expected payoff from two-period game is

$$2I_1^i/x_1 - I_1^i + \delta(1 - (I_1^1 + I_1^2)/x_1)V_1(x_1 - I_1^1 - I_1^2),$$

where player i incurs searching cost of I_1^i, gets immediate expected payoff of $2I_1^i/x_1$, and anticipates the expected payoff of $\delta(1-(I_1^1+I_1^2)/x_1)V_1(x_1-I_1^1-I_1^2)$ in the second period. Simplifying the above expression yields

$$I_1^i + \frac{\delta}{2}(1 - I_1^1 - I_1^2)(1 + I_1^1 + I_1^2).$$

The first-order conditions with respect to I_1^i are

$$1 - \delta(I_1^1 + I_1^2) = 0.$$

Whenever $I_1^1 + I_1^2 \leq x_1 = 1$ and provided there is no duplication, player i has an incentive to search as much as possible in the first period. Consequently, there is a unique symmetric SPE that involves each player finishing the search in one period by searching a complementary half of the island; that is, $I_1^1 = I_1^2 = x_1/2 = 1/2$. Player i's value in the two-period game with the island size of $x_1 = 1$ is $\hat{V}_2(x_1) = \frac{1}{2}$. The total value of the game for two players is $2\hat{V}_2 = 1$. On the other hand, the total value of the game for one player is $V_2 = \frac{13}{12}$. Thus, two players search too fast. This is what we refer to as the *tragedy of the commons* effect.

3 The Model

In the model, $n \geq 1$ players are searching for a treasure hidden somewhere on an island.[9] The treasure has the same value $R > 0$ for all players, and there is an equal

[9]The case when there is a positive probability that the treasure does not exist is considered in Section 5.

probability that the treasure will be located at any given point on the island.[10] We assume that each player observes how much the other players have searched previously before making his search plans. Denote the area player i searches in period t as I_t^i. Let the unsearched area of the island at period t be given by x_t. The size of the island is assumed to be $x_1 > 0$.[11]

At period $t = 1$ all players observe x_1 and simultaneously choose how much to search for the treasure. The search is costly. If player i searches I_1^i, his search cost is $-cI_1^i$, where $c > 0$.[12] It is assumed that if players search the same area and find the treasure simultaneously, each of them incurs costs, but the treasure will be destroyed.[13] Consequently, in equilibrium players never search the same area. Altogether in period 1 players search an area of

$$J_1 = I_1^1 + ... + I_1^n.$$

Note that given players never search the same area and searching outside the island has no benefit, it must be the case that $J_1 \leq x_1$.

If $J_1 = x_1$, player i has a I_1^i/x_1 probability of finding the treasure, and the game ends. Player i obtains the following expected payoff:

$$\frac{I_1^i}{x_1} R - cI_1^i.$$

If $J_1 < x_1$, player i has a I_1^i/x_1 probability of finding the treasure, and the game ends with probability J_1/x_1. If the treasure is not found at period $t = 1$ (this happens with probability $1 - J_1/x_1$), the unsearched area of the island shrinks to $x_2 = x_1 - J_1$, and the game proceeds to the next period $t = 2$.

At period $t > 1$, each player knows the *history* $h_t = (x_1; J_1, ..., J_{t-1})$ and all players simultaneously choose how much to search for the treasure on the previously unsearched area of size x_t. Given that players never search the same area and searching the previously searched area has no benefit, it must be the case that $J_t \leq x_t$.

If $J_t = x_t$, player i has a I_t^i/x_t probability of finding the treasure, and the game ends with probability one. The expected payoff for player i in period 1 for period t is

$$\delta^t \frac{I_t^i}{x_t} R - c(I_1^i + \delta I_2^i + ... + \delta^t I_t^i),$$

where δ is the common discount factor.

If $J_t < x_t$, player i has a I_t^i/x_t probability of finding the treasure, and the game ends with probability J_t/x_t. If the treasure is not found at period t (this happens

[10]We focus our attention on uniform distribution because this is the most realistic assumption when there is no information about the island. See Section 6 for further discussion.

[11]In Appendix A we provide two alternative interpretations of the model presented in this section.

[12]Non-linear costs are considered in Section 5.

[13]See Section 5 for an alternative assumption.

with probability $1 - J_t/x_t$), the unsearched area of the island shrinks to

$$x_{t+1} = x_t - J_t.$$

The new unsearched area is equal to the previous one, minus the searched part. Note that all search costs are sunk, but only one player (if any) can find the treasure. Moreover, the value of the prize is known from the beginning, but the search costs for each player will be determined only at the end of the game.

Player i's strategy is an infinite sequence of functions specifying how much to search each period contingent upon any possible sequence of previous searches. The game we consider is stochastic, and any history can be summarized by the "state", the current unsearched area. We will consider Markov strategies in which the past influences the current play only through its effect on the current unsearched area. A pure Markov strategy for player i is a time-invariant map $I^i : X \to X$, where $X = [0, x_1]$. We will also restrict our attention to symmetric equilibria. Therefore, the solution concept is a *symmetric Markov perfect equilibrium* (SMPE).[14] Moreover, because in general there exist multiple SMPE, we focus on the Superior SMPE (SSMPE); that is, the SMPE with the highest total expected payoff in the absence of collusion.[15]

We use the following approach to obtaining the SSMPE. Player i takes the state-contingent search plans of his rivals as given; these plans can be aggregated to obtain $I^{-i}(x)$. Given this function, player i solves a standard optimization problem and chooses his optimal search, $I^i(x)$. However, given that the function $I^{-i}(x)$ is endogenous to the model, to obtain the symmetric equilibrium we need to find the function $I^{-i}(x)$ such that $(n-1)I^i(x) \equiv I^{-i}(x)$, where $I^i(x)$ is the optimal search for player i when he takes I^{-i} as given.

Player i solves the following Bellman equation:

$$V(x, c, R) = \max_{I^i \in [0, x - I^{-i}]} \{-cI^i + \frac{I^i}{x} R + \delta \left(1 - \frac{I^i + I^{-i}}{x}\right) V(x - I^i - I^{-i}, c, R)\}, \quad (1)$$

where x is the part of the island which is still unsearched before the current period, $V(x, c, R)$ is the value function for each player (we use the symmetry assumption here) and the solution has to satisfy $(n-1)I^i = I^{-i}$. The first term in equation (1) describes the player's costs of search in the current period. The second term is the player's expected value from finding the treasure in the current period. The last term is the player's expected value from future periods.

4 Analysis of the Model

In this section we derive a general procedure for finding the SSMPE. Note that x, I^i, and I^{-i} are of the same unit measure. To simplify the analysis, and work with unit

[14] We discuss both subgame perfect and asymmetric Markov perfect equilibria in Section 6.

[15] As argued by Fudenberg and Tirole (1985), if one equilibrium Pareto dominates all others, it is the most reasonable outcome to expect.

free variables, we introduce the following lemma.

Lemma 1.

$$V(x, c, R)/R = V(cx/R, 1, 1). \tag{2}$$

Proof. See Appendix B.

Slightly abusing notation, rename $V(x) = V(x, 1, 1)$. From Lemma 1 it follows that it is sufficient to solve equation (1) when $R = c = 1$; that is,

$$V(x) = \max_{I \in [0, x - I^{-i}]} \{(1 - x)I/x + \delta(x - I - I^{-i})V(x - I - I^{-i})/x\}, \tag{3}$$

where $I = I^i$ for simplicity. Note that an increase in cost parameter c in the original problem (1) is equivalent to an increase in the unsearched area size x in problem (3). On the other hand, an increase in the treasure value R has two effects; it decreases the unsearched area size x, but it also directly changes the value function. We discuss comparative statics with respect to x, c and R later in this section.

Let us concentrate on problem (3). To rule out cases where no investment is made in equilibrium we introduce the following definition.

Definition 1. *A SMPE is trivial if* $I_t = 0 \ \forall \ t$.

We want to focus our attention on SMPE that are non-trivial. To simplify the exposition, it is convenient to introduce the following function:

$$\Psi(x) \equiv xV(x). \tag{4}$$

From definition (4), it follows that

$$\Psi(x) \geq 0 \ \text{ for any } \ x.$$

Note that $\Psi(x)$ can be interpreted in a similar way as expected value; it transforms a problem involving probabilities into a deterministic one. It also reframes the problem providing an alternative interpretation to the original model; see Appendix A.

In terms of $\Psi(x)$ equation (3) can be rewritten as

$$\Psi(x) = \max_{I \in [0, x - I^{-i}]} \{(1 - x)I + \delta\Psi(x - I - I^{-i})\}, \tag{5}$$

where the solution has to satisfy $I^{-i} = (n - 1)I$.

Before proceeding, we introduce the following operator B on the set of continuous functions

$$(B\Psi)(x) \equiv \max_{I \in [0, x - I^{-i}]} \{(1 - x)I + \delta\Psi(x - I - I^{-i})\}, \tag{6}$$

where, again, the solution has to satisfy $I^{-i} = (n - 1)I$.

Lemma 2.

(a) Any non-trivial SMPE involves $I_t > 0$ *for* $0 \leq t \leq T$*, and* $I_t = 0$ *for* $t > T$*, where*

T is finite.
(b) All SMPE can be obtained in a finite number of steps. Specifically, there exists k such that

$$\Psi_0 \equiv 0, \qquad \Psi_i \equiv B\Psi_{i-1} \qquad \textit{for} \quad i = 1, 2, \ldots, k, \qquad \textit{and} \qquad \Psi = \Psi_k.$$

Proof. See Appendix B.

Note that with the help of Lemma 2, one can construct the sequence $\{\Psi_i\}$ and find all SMPE. This is called the value-iteration procedure. This procedure is equivalent to using backward induction. In general, there will be multiple SMPE. As discussed earlier, we focus on the unique SSMPE.

4.1 Monopoly $n = 1$

In this section we give a complete characterization of the unique solution to problem (3) when $n = 1$. Let us directly apply Lemma 2 to derive the following proposition.

Proposition 1. *When $n = 1$, the unique solution to problem (3) is described as follows. The search intensity is*

$$I(x) = \begin{cases} x, & \textit{if} \quad x \le t_1, \\ \frac{(2\delta-1)(x-1)+\delta}{2\delta}, & \textit{if} \quad t_1 < x \le t_2, \\ \vdots & \\ (x-1)(1 - \frac{\sin(i-1)\varphi}{\delta^{1/2}\sin i\varphi}) + \frac{\delta^{(i-1)/2}\sin\varphi}{\sin i\varphi}, & \textit{if} \quad t_{i-1} < x \le t_i, \\ \vdots & \\ (x-1)(1 - \frac{\sin(m-1)\varphi}{\delta^{1/2}\sin m\varphi}) + \frac{\delta^{(m-1)/2}\sin\varphi}{\sin m\varphi}, & \textit{if} \quad t_{m-1} < x \le u_m, \\ 0, & \textit{if} \quad x > u_m; \end{cases}$$

and the value function is[16]

$$V(x) = \Psi(x)/x = \begin{cases} P_1(x)/x, & \textit{if} \quad x \le t_1, \\ \vdots & \\ P_m(x)/x, & \textit{if} \quad t_{m-1} < x \le u_m, \\ 0, & \textit{if} \quad x > u_m; \end{cases}$$

$$P_i(x) = -\frac{\sin(i+1)\varphi}{2\sin i\varphi}\delta^{-1/2}(1-x)^2 + \frac{\sin\varphi}{\sin i\varphi}\delta^{(i-1)/2}(1-x) + \frac{\sin(i-1)\varphi}{2\sin i\varphi}\delta^{(2i-1)/2} \quad \textit{for } i \in [1, m],$$

$$t_i = 1 - \delta^{i/2}\cos i\varphi, \quad u_m = 1 + \frac{\delta^{m/2}(\sin m\varphi - \sin\varphi)}{\sin(m+1)\varphi}, \quad \varphi = \arccos\sqrt{\delta}, \quad m = \lceil \pi/\varphi - 1 \rceil.$$

[16]Note that $\sin i\varphi = 0$ for some $i \in [1, m]$ is not possible because it implies that the optimal sequence includes $i - 1$ steps only; for details see Appendix B.

Proof. See Appendix B.

The optimal strategy for the monopolist is to plan to search the island for at most m periods. In general there are $m+1$ possibilities, depending on the island size. The first possibility is that the monopolist searches the island for just one period. Intuitively this happens for small values of $x \leq t_1$; a small island induces the monopolist to search the whole island immediately as it is too costly to endure a probability of having to wait an extra period to gain the prize. The second possibility is that the firm plans to search the island for at most two periods. This happens for larger values of $x \in (t_1, t_2]$. In general, the monopolist plans to search the island for at most i periods for values of $x \in (t_{i-1}, t_i]$, where $i = 1, \ldots, m-1$. Furthermore, the firm plans to search the island for at most m periods when $x \in (t_{m-1}, u_m]$. The monopolist can also find search to be too costly, and abstain from searching at all. This happens for large values of $x > u_m$. Overall, this means the optimal search intensity $I(x)$ is a spline of degree one on the interval $[0, u_m]$ with knots $t_1, ..., t_{m-1}$.[17] Let us characterize the optimal search intensity $I(x)$ derived in Proposition 1 in the following corollary.

Corollary 1. *When $n = 1$ and $x \leq u_m$, the optimal search intensity $I(x)$ is a piecewise linear, continuous and concave function.*

Proof. See Appendix B.

Figure 1a illustrates the optimal search intensity $I(x)$ for the monopolist when $\delta = 0.75$. In this case $\varphi = \pi/6$ and $m = 5$. Observe that conditional on a project being financed in 5 periods, in the first 3 periods the optimal search intensity is increasing over time. On the other hand, starting from period 3, the optimal search intensity decreases. The intuition for the initial increase in intensity is that when the unsearched area is relatively large, the value that the monopolist derives from search is relatively small, so that it only invests a small amount. Once the unsearched area shrinks, the value of search rises, which leads to a larger amount to be invested. On the other hand, when the unsearched area is very small, investing a large amount is costly because of the potential for overshooting. The optimal search intensity starts decreasing as a result, explaining the decline in periods 4 and 5.

Now we characterize the value function $V(x)$ derived in Proposition 1 in the following corollary.

Corollary 2. *When $n = 1$ and $x \leq u_m$, the value function $V(x)$ is continuously differentiable and monotonically decreasing.*

Proof. See Appendix B.

[17] A spline is a special function defined piecewise by polynomials, see for example Ahlberg, Nielson, and Walsh (1967). The number of subintervals of the spline varies from $m = 2$ when $\delta < 1/4$, to $m \to \infty$ as $\delta \to 1$.

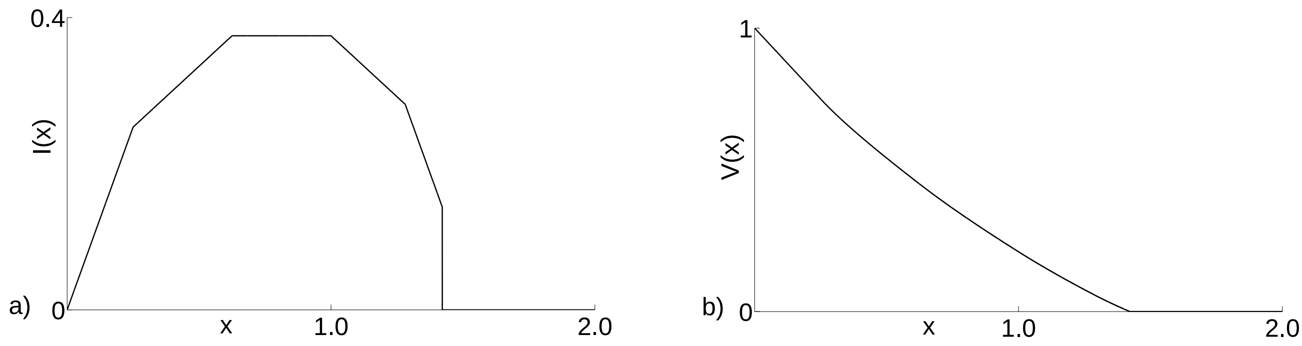

Figure 1: The optimal search intensity and value for the monopolist when $\delta = 0.75$.

Figure 1b illustrates the value function when $\delta = 0.75$. Observe that the value function is a smooth monotonically decreasing function. This is an intuitive result. On the one hand, the value is monotonically decreasing because it is harder to find the treasure on a larger island. On the other hand, the value function is smooth because the strategy of the monopolist is unimpeded by the constraints potentially imposed by other competitors. As we show later, neither monotonicity not smoothness are always satisfied with multiple firms. We turn to this issue now.

4.2 Duopoly $n = 2$

In this section we give a complete characterization of SSMPE when $n = 2$. We directly apply Lemma 2 to derive:

Proposition 2. *When $n = 2$, the non-trivial SSMPE exists if and only if $x \leq u_d$ and is described as follows. The search intensity of each firm is*

$$I(x) = \begin{cases} \frac{x}{2}, & \text{if} \quad x \leq t_1, \\ \frac{2-\delta-2(1-\delta)x}{4\delta}, & \text{if} \quad t_1 < x \leq t_2, \\ \vdots & \\ \frac{2-2\delta+\delta/2^{i-2}-2(1-\delta)x}{4\delta}, & \text{if} \quad t_{i-1} < x \leq t_i, \\ \vdots & \\ \frac{2-2\delta+\delta/2^{d-2}-2(1-\delta)x}{4\delta}, & \text{if} \quad t_{d-1} < x \leq u_d; \end{cases}$$

and the value function for each firm is[18]

$$V(x) = \Psi(x)/x = \begin{cases} P_1(x)/x, & \text{if} \quad x \leq t_1, \\ \vdots & \\ P_d(x)/x, & \text{if} \quad t_{d-1} < x \leq u_d; \end{cases}$$

[18]Some of the expressions below are presented as ratios for expositional purposes. In particular, note that both numerators and denominators in P_i, t_i, u_i and d contain either $(1 - 4\delta)$ or $\ln(4\delta)$ term. When $\delta = 1/4$, these expressions are defined as their limits as $\delta \to 1/4$.

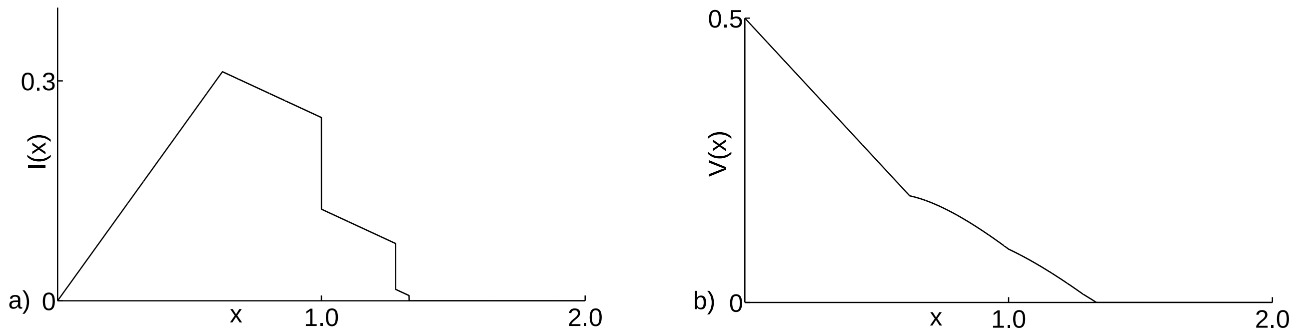

Figure 2: The equilibrium search intensity and value for duopolists when $\delta = 0.75$.

$$P_i(x) = -\frac{1}{2}(1-x)^2 + \frac{1}{2^i}(1-x) + \left(\frac{(4\delta)^{i-1}-1}{4^{i-1}(4\delta-1)}\right)\frac{\delta}{2} \ \ for\ i \in [1, d],$$

$$t_i = 1 - \frac{3\delta + (4\delta)^i(\delta-1)}{2^i(4\delta-1)}, u_d = 1 + \frac{1}{2^d}\left(\sqrt{\frac{1-(4\delta)^d}{1-4\delta}} - 1\right), d = \left\lceil \frac{\ln\left((\delta^2+2\delta)/(1-\delta)^2\right)}{\ln(4\delta)} \right\rceil.$$

Proof. See Appendix B.

In the non-trivial SSMPE duopolists plan to search the island for at most d periods. There are d possibilities, depending on the island size. The first possibility is that the duopolists search the island for just one period when $x \leq t_1$. The second possibility is that the firms plan to search the island for at most two periods when $x \in (t_1, t_2]$. In general, the duopolists plan to search the island for at most i periods for values of $x \in (t_{i-1}, t_i]$, where $i = 1, \ldots, d-1$. Furthermore, the firms plan to search the island for at most d periods when $x \in (t_{d-1}, u_d]$. Note that an important difference from the monopoly case is that no SMPE exists for $x \in (u_d, u_m)$.[19] Finally, when $x \geq u_m$ the unique SMPE is the trivial one.

As one can see from Proposition 2, the equilibrium search intensity $I(x)$ is a spline of degree one on the interval $[0, u_d]$ with knots $t_1, ..., t_{d-1}$. Let us characterize the equilibrium search intensity in the following corollary.

Corollary 3. *When $n = 2$ and $x \leq u_d$, the equilibrium search intensity $I(x)$ is a piece-wise linear, discontinuous and quasiconcave function.*

Proof. See Appendix B.

Figure 2a illustrates the equilibrium search intensity in the SSMPE for each duopolist when $\delta = 0.75$. In this case $d = 4$. Observe that, conditional on the project being financed in 4 periods, the equilibrium search intensity is increasing in

[19]There are asymmetric and mixed strategy equilibria when $x \in (u_d, u_m)$. This is akin to two firms considering entering a market that is a natural monopoly, Bolton and Farrell (1990). Given that our focus is on the SSMPE, we do not derive these equilibria here. See, for example, Smirnov and Wait (2007) for the analysis of a similar problem.

the first 3 periods. In the last period the equilibrium search intensity decreases. This is due to the fact that, while in the last period the duopolists search all of the remaining land, this area can be smaller than the area searched in the second-to-last period. Note that the equilibrium search intensity is a discontinuous function. At knots of the spline t_i, the equilibrium strategy of a firm switches from searching for i periods to searching for $i+1$ periods. The fact that each firm maximizes its own value, not total surplus, leads to this discontinuity. We discuss this difference in incentives in more detail in Section 4.3.

The following corollary characterizes the value function $V(x)$, as derived in Proposition 2.

Corollary 4. *When $n = 2$ and $x \leq u_d$, the value function $V(x)$ is a continuous and monotonically decreasing function.*

Proof. See Appendix B.

Figure 2b illustrates the value function when $\delta = 0.75$. Observe that, consistent with the monopoly case, the value function is monotonically decreasing with the size of the unsearched area because it is harder to find the treasure on a larger island. However, contrary to the monopoly case, the value function is not smooth; the slope of the value function changes at knots t_i, coinciding with the discontinuities in the equilibrium search intensity.

As the duopolists maximize their own payoff rather than overall surplus, their search can create negative externalities. We examine this issue in the next section.

4.3 Tragedy of the commons and free riding in search

In previous sections we have described the equilibria for the cases of monopoly and duopoly. This puts us in a position to compare how two-player search differs from the search strategy of the benevolent planner (monopolist).

Consider equation (5):

$$\Psi(x) = \max_{I \in [0, x - I^{-i}]} \{(1 - x)I + \delta\Psi(x - I - I^{-i})\}.$$

We observe two types of inefficiencies in the duopoly model: over-investment (tragedy of the commons) and under-investment (free riding). Over-investment can occur when the value of x is small, $0 < x < 1$. In this case the current payoff, the first term in equation (5), is positive; that is, $(1 - x)I > 0$. In a duopoly case, each player has incentives to increase his current expected payoff by searching more. In equilibrium players oversearch, which is similar to a standard tragedy of the commons effect. Under-investment arises for larger values of x, namely when $x > 1$. In this case the current payoff is negative, so each player has an incentive to decrease his current losses by undertaking less search. This is similar to a standard free-riding effect.

Figure 3 illustrates these two cases: tragedy of the commons (for $x < 1$) and free riding (for $x > 1$). For small values of x, aggregate search $J(x)$ is the same for $n = 1$

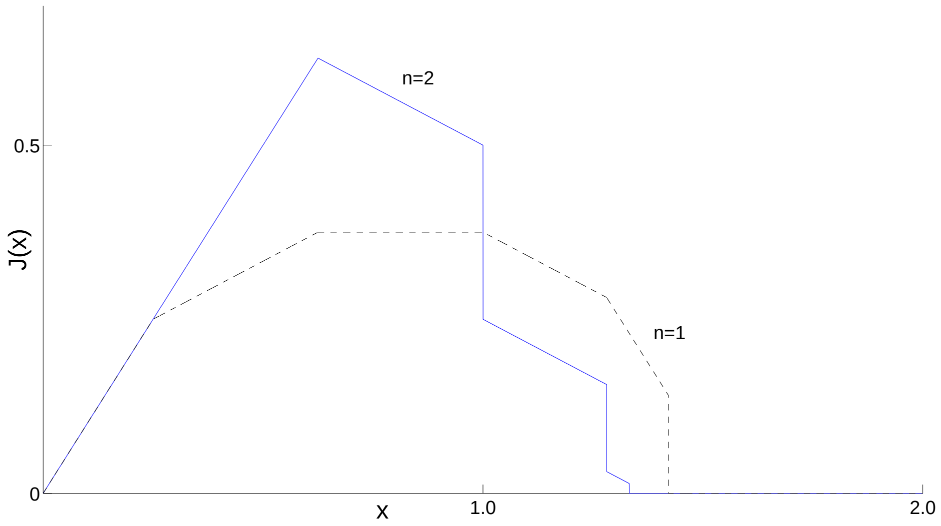

Figure 3: $J(x)$: $n = 1$ is a dashed line, $n = 2$ is a solid line; $\delta = 0.75$.

and $n = 2$; players search the whole island in just one period. On this part of the graph, both curves coincide. For larger values of x, but still less than one, the curve for $n = 2$ is above the curve for $n = 1$. This means that the monopolist, $n = 1$, searches the island efficiently, while in the case of two players, $n = 2$, the tragedy of the commons effect takes place; players search the whole island too fast. Finally, we can see that the curve for $n = 1$ is above the curve for $n = 2$ when $x > 1$. Free riding takes place on this interval, and two players search the island too slowly relative to the monopolist.

It worth noting that with one or two players, despite the existence of negative externality in a duopoly, the value function $V(x)$ is monotonically decreasing. From this monotonicity and equation (2), it follows that $V(x, c, R)$ is monotonically decreasing with respect to x and c and monotonically increasing with respect to R. These monotonicity results, however, do not necessarily hold with more than two players, as we discuss in the next section.

4.4 Three or More Players

As we showed above, $V(x)$ is always monotonically decreasing for $n = 1, 2$. Let us show that this is not always the case. To do this, first, consider when $n \geq 3$ players search for at most two periods in equilibrium.[20]

Proposition 3. *When $n \geq 3$, the SSMPE for $x \leq \min[t_2, u_2]$ is described as follows.*

[20]The focus here is to show non-monotonicity in the value function. Deriving the SSMPE for more than two periods is not necessary to demonstrate this result and, moreover, it can be very tedious.

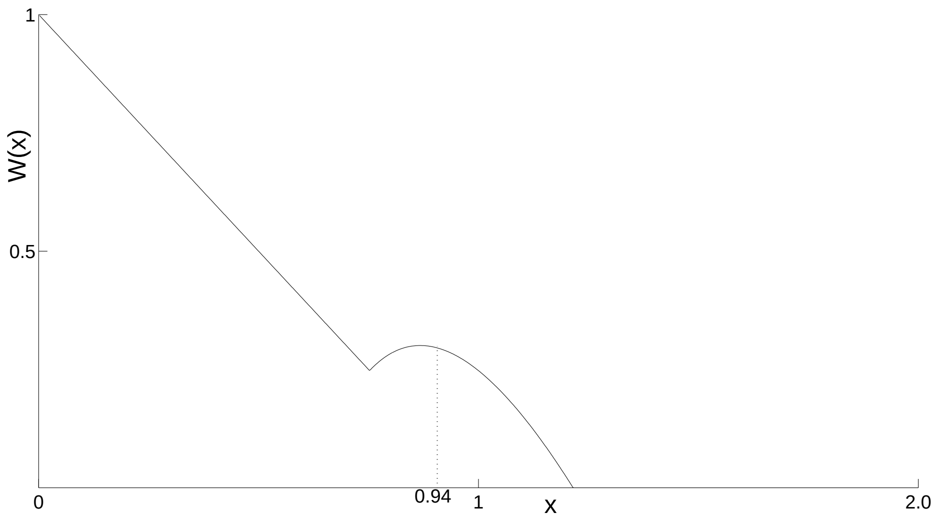

Figure 4: $W_4(x) = 4V(x)$.

The equilibrium search intensity is

$$I(x) = \begin{cases} \frac{x}{n}, & \textit{if} \quad x \le t_1, \\ \frac{(2\delta-n)x+n-\delta}{2\delta n}, & \textit{if} \quad t_1 < x \le \min[t_2, u_2]; \end{cases}$$

and the value function is

$$V(x) = \Psi(x)/x = \begin{cases} (1-x)/n, & \textit{if} \quad x \le t_1, \\ \frac{-(4\delta+n^2-2n)(1-x)^2+2\delta(1-x)+\delta^2}{4\delta n x}, & \textit{if} \quad t_1 < x \le \min[t_2, u_2]; \end{cases}$$

$$t_1 = 1 - \frac{\delta}{n}, \;\; t_2 = 1 + \frac{-(1+2s) + \sqrt{(1+2s)^2 - 4s^2\delta(1-4(1-\delta)(1+s))}}{8s^2},$$

$$u_2 = 1 + \frac{\sqrt{4\delta(s+1)+1} - 1}{4(s+1)}, \;\; \textit{and} \;\; s = \frac{n(n-2)}{4\delta}.$$

Proof. See Appendix B.

Now we apply this technique to the following example. Suppose that $n = 4$ and $\delta = 0.99$. Using Proposition 3, we derive the following value function

$$V(x) = \begin{cases} (1-x)/4, & \text{if} \quad x \le 0.7525, \\ (-3.0202(1-x)^2 + 0.5(1-x) + 0.2475)/(4x), & \text{if} \quad 0.7525 < x \le 0.94. \end{cases} \tag{7}$$

Figure 4 illustrates the combined four-player value function ($W_4(x) = 4V(x)$) for $n = 4$ and $\delta = 0.99$. Let us consider this value function $W_4(x)$ when $x \le 0.94$. It would seem natural to expect that a larger island will make players worse off.

However, consider two island sizes

$$\widetilde{x} = 0.75251 < 0.7526 = \widehat{x}.$$

Using (7), we get

$$V(0.75251) = 4.6563 \times 10^{-2} < 4.6586 \times 10^{-2} = V(0.7526).$$

In this case, a larger island makes all players better off. The following proposition demonstrates that this non-monotonicity result holds whenever there are at least three players.

Proposition 4. *When $n \geq 3$, V(x) is not monotonically decreasing function; specifically, there exists $x > t_1$ such that $V(x) > V(t_1)$.*

Proof. See Appendix B.

This surprising observation can be explained by inefficient player behavior when x is relatively small. Due to the tragedy of the commons effect, players search the smaller $\widetilde{x}$ too fast in the first period, and leave too little for second period search. With a slightly larger island of $\widehat{x}$, players search the island more slowly and efficiently (more closely replicating the search pattern of the monopolist). When there is a sufficiently large number of players (here $n = 4$), the efficiency improvement from mitigating the over-investment problem can outweigh the additional expected cost of searching a larger island.

Given non-monotonicity of $V(x)$, $V(x, c, R)$ is not monotonic in x. In addition, from non-monotonicity of $V(x)$ and equation (2), it follows that $V(x, c, R)$ does not monotonically decrease with respect to c. We summarize these results in the following corollary.

Corollary 5. *When $n \geq 3$, $V(x, c, R)$ is not a monotonically decreasing function with respect to either x or c; specifically, there exists $\tilde{x} > x$ such that $V(\tilde{x}, c, R) > V(x, c, R)$ and there exists $\tilde{c} > c$ such that $V(x, \tilde{c}, R) > V(x, c, R)$.*

The comparative statics with respect to R do not automatically follow from Proposition 4. Rather, from Lemma 1, $V(x, c, R) = V(cx/R, 1, 1)R$, so a change in R could have two opposing effects. However, the following proposition shows that there is a non-monotonicity in this case as well.

Proposition 5. *When $n \geq 4$, $V(x, c, R)$ is not a monotonically increasing function with respect to R; specifically, there exists $\tilde{R} > R$ such that $V(x, c, \tilde{R}) < V(x, c, R)$.*

Proof. See Appendix B.

Note that, contrary to previous results, Proposition 5 requires at least four firms. To outweigh the increase in the value R, in addition to the decrease in the cost of searching the smaller island cx/R, a larger number of firms are required.

We conclude this section with an investigation of multi-player search efficiency.

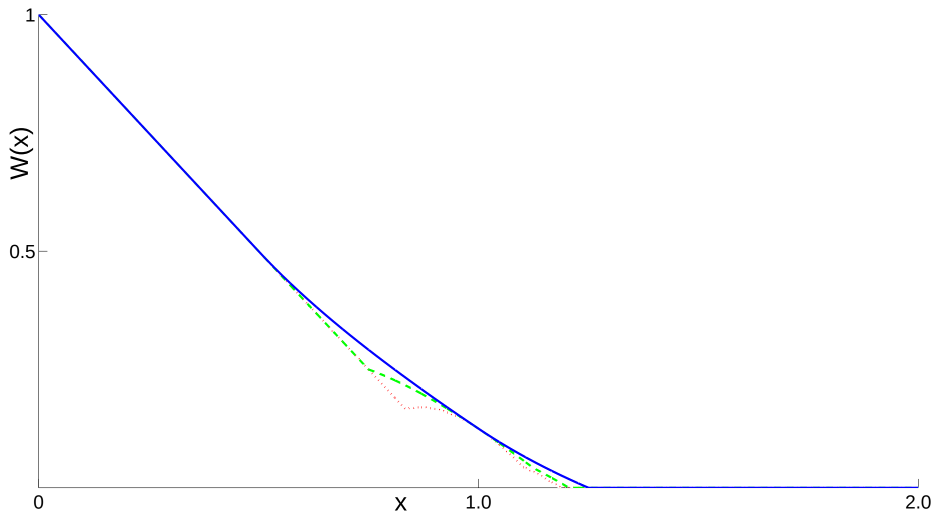

Figure 5: $W_1(x)$ is a solid line, $W_2(x)$ is a dashed line, $W_3(x)$ is a dotted line; $\delta = 0.5$.

4.5 Efficient search

In this section we compare a monopolist's strategy to search intensities with many players. Given it captures all of the surplus the monopolist in our model always searches efficiently. It is also intuitive that the multi-player search reproduces the monopoly outcome when the island is small; players search the whole island in just one period. However, efficient search is not limited only to small islands. If the combined n-player value function is defined as $W_n(x) = nV(x)$, then

Proposition 6. *When* $0 < \delta \leq 0.5$, $W_n(1) = W_1(1)$ *for any* $n \geq 2$.

Proof. See Appendix B.

Figure 5 illustrates Proposition 6. It shows the combined value function when one, two, or three players search the island and $\delta = 0.5$. Note that for $\delta \leq 0.5$ and $x \leq 1$, the project is completed in at most two periods, which means that the constructed equilibrium for $x \leq 1$ is the SSMPE. Furthermore, the total value function at $x = 1$ is the same in all three cases. When $x = 1$, neither the tragedy of the commons nor the free-riding effect is present; as a result, the multi-player search reproduces the monopoly outcome. This result has the following intuition. For this unique island size, players get zero expected payoff in period one of the two-period search. Consequently, the objective function of the players is to maximize their expected payoffs from the second period only. In the second period of the symmetric equilibrium, each firm receives a payoff proportional to the payoff of the monopoly. This guarantees that multi-player search reproduces the monopoly outcome.

5 Extensions

5.1 Uncertainty about treasure existence

In this section, we consider the case when there is a positive probability that the treasure does not exist. One possible interpretation of this situation is when the treasure is located on the island for sure but players may search only some part of the island.

If the probability that the treasure exists on an island of size x' is $\pi > 0$, then the island can be extended to a size of $x = x'/\pi$ with a convention that the treasure belongs to the island of size x for sure. However, only the portion x' of that island can be searched, while the portion $x - x'$ is not accessible to the players. We denote the portion of island unaccessible by players, $x - x'$, as α.

Using this new structure, equation (3) transforms to

$$V(x) = \max_{I \in [0, x - I^{-i} - \alpha]} \{(1-x)I/x + \delta(x - I - I^{-i})V(x - I - I^{-i})/x\}, \tag{8}$$

and the solution has to satisfy $I^{-i} = (n-1)I$. The only difference from the main model is that $\alpha > 0$ represents the area of the island that players cannot search.[21] This means the unsearched area of the island, x, includes both regions of the island that players can and cannot search. Consequently, the portion of the island that all players search in the current period has to satisfy $I + I^{-i} \leq x - \alpha$. Note that this new model still has the property that searching becomes more attractive over time. As a result, in any non-trivial equilibrium, players will continue to search until they either find the treasure or have searched the entire accessible region.

As in the main model, it is convenient to introduce the following function $\Psi(x) \equiv xV(x)$. Equation (8) transforms to

$$\Psi(x) = \max_{I \in [0, x - I^{-i} - \alpha]} \{(1-x)I + \delta\Psi(x - I - I^{-i})\}, \tag{9}$$

where as earlier the solution has to satisfy $I^{-i} = (n-1)I$.

Next we show that the problem with $\alpha > 0$ can be transformed to the original problem with $\alpha = 0$.

Proposition 7. *Substitutions $x' = \frac{x-\alpha}{1-\alpha}$ and $\Psi' = \frac{\Psi}{(1-\alpha)^2}$ transform problem (9) into problem (5).*

Proof. See Appendix B.

Figure 6 illustrates the equilibrium search intensity and the value function for duopolists when $\delta = 0.75$ and $\alpha = 0.5$. As one can see, the substitution $x' = 2x - 1$ transforms Figure 6a to Figure 2a. This means problem (9) can be derived from (5) by a linear transformation. The intuition for this is that both (5) and (9) represent

[21] If $\alpha = 0$, we are back in the main model.

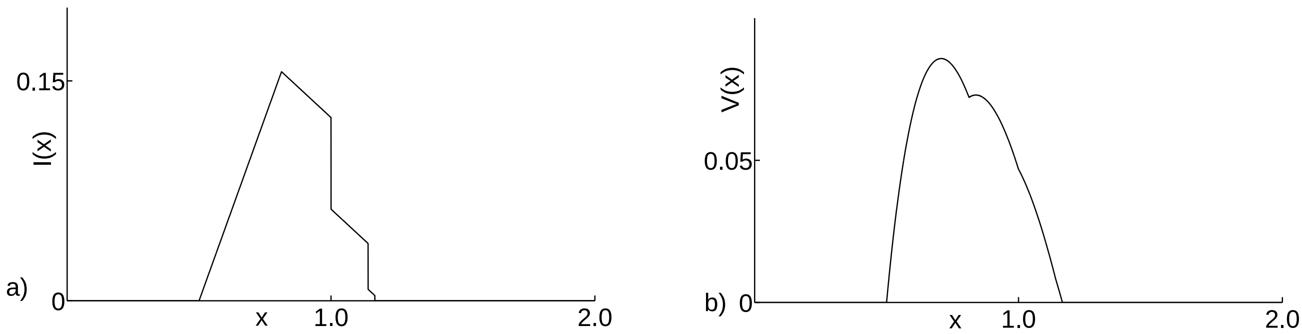

Figure 6: SSMPE for duopolists when $\delta = 0.75$ and $\alpha = 0.5$.

the multi-player cake eating problem, described in the Introduction and Appendix A. In this game, each player's strategy depends only on the remaining cake size (be it x or $x - \alpha$), making both problems isomorphic.

Figure 6b illustrates the value function for duopolists when $\delta = 0.75$ and $\alpha = 0.5$; it is very different from Figure 2b. The transformation from Ψ back to the original value function creates an additional effect. When $x \approx \alpha$, an increase in x leads to a significant increase in the probability that the treasure is located on the part of the island that can be searched. This leads to a sharp increase in value, as shown in Figure 6b. However, once the island is significantly larger than α, further increases in x have a relatively small impact on the probability that the treasure belongs to the searchable part of the island. In this case, it is the increase in search costs that have the larger effect, meaning that value declines in x. Consequently, there is a non-monotonic relationship between the value and island size.

5.2 Convex costs

Let us modify the main model by examining convex costs, specifically $C(x) = x^2$.[22] Here, player i solves the following Bellman equation:

$$V(x) = \max_{I^i \in [0, x - I^{-i}]} \{-(I^i)^2 + R\frac{I^i}{x} + \delta \left(1 - \frac{I^i + I^{-i}}{x}\right) V(x - I^i - I^{-i})\}, \qquad (10)$$

where $V(x)$ is the value function for each player and the solution has to satisfy $(n-1)I^i = I^{-i}$. The first term in equation (10) describes the player's costs of search in the current period. The second term is the player's expected value from finding the treasure in the current period. The last term is the player's expected value from future periods.

The following specific example provides us with the key intuition for this extension. Suppose that $R = 1$, $n = 2$, $\delta = 0.9$ and $x_1 = 1$. The SSMPE for $x \leq 1$ is described

[22] $C(x) = ax + x^2$ with $a > 0$ produces qualitatively similar results.

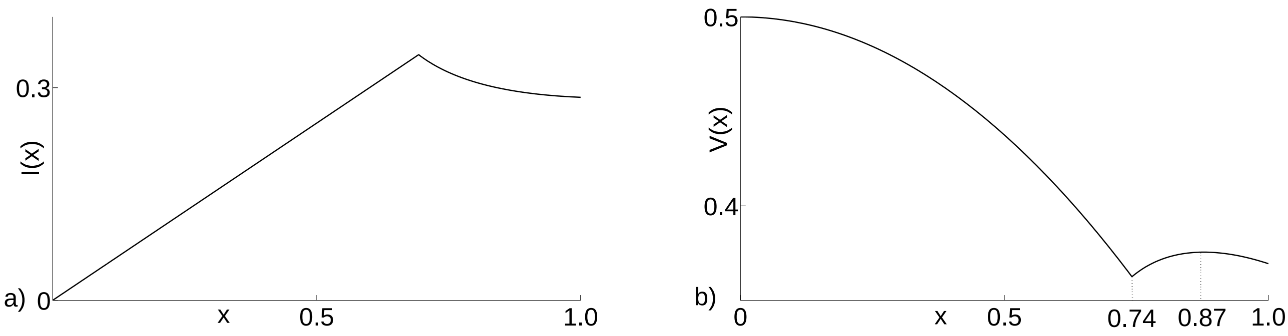

Figure 7: The equilibrium search intensity and value function for duopolists when $C(x) = x^2$.

as follows.[23] The equilibrium search intensity is

$$I(x) = \begin{cases} x/2, & \text{if} \quad x \leq 0.74, \\ \frac{4.7-\sqrt{14.8x^2-5.94}}{5.4}, & \text{if} \quad 0.74 < x \leq 1; \end{cases}$$

and the value function is

$$V(x) = \begin{cases} (2-x^2)/4, & \text{if} \quad 0 \leq x \leq 0.74, \\ \left[4I(x)(1-xI(x)) + 0.9(x-2I(x))(2-(x-2I(x))^2\right]/(4x), & \text{if} \quad 0.74 < x \leq 1. \end{cases}$$

Figure 7 illustrates the equilibrium search intensity $I(x)$ and the value function $V(x)$ for this example. As we can see from Figure 7a, even though it has a similar structure to the equilibrium search intensity in the main model, the equilibrium search intensity here is not piece-wise linear.

Figure 7b illustrates the value function, which is non-monotonic in x; for example, $V(0.87) > V(0.74)$. In contrast to the model with linear costs, quadratic costs generate a non-monotonicity of the value function in a duopoly setting. Similar to the linear model, duopolists with convex costs search the smaller island too fast in the first period, and leave too little for second period search. If the size of the island is increased, duopolists search the island more slowly and efficiently. This efficiency improvement as x increases could be larger with quadratic costs than with linear costs; this improvement, even with two players, can outweigh the costs of search a larger island, generating the non-monotonicity in value.

Note also that the assumption about linear costs in the main model not only greatly simplifies the analysis but also guarantees that the single-player problem coincides with that of a benevolent planner. With convex costs (increasing marginal costs) there is an additional effect that favors multi-player search; this complicates the analysis of the tragedy of the commons and free-rider externalities.

[23]See Appendix B for derivations.

5.3 Duplication

Now consider when players cannot avoid search duplication. We also assume that if players find the treasure simultaneously, they share it equally (rather than each receiving no prize). We restrict our attention to $n = 2$ players and $R = 1$. Given the unsearched area x, each player i chooses search intensity $I^i \in [0, x]$; search is random so that each point on $[0, x]$ is selected with equal probability. Equation (3) transforms to

$$V(x) = \max_{I\in[0,x]}\{-I + \frac{I}{x}\left(1 - \frac{I^{-i}}{x}\right) + \frac{I}{2x}\left(\frac{I^{-i}}{x}\right) + \delta\left(1 - \frac{I}{x}\right)\left(1 - \frac{I^{-i}}{x}\right)V(x - I)\}, \tag{11}$$

where, as earlier, $V(x)$ is the value function for each player and the solution has to satisfy $I^{-i} = I$. The first term in equation (11) describes the player's costs of search in the current period. The second term is the player's expected value from finding the treasure in the current period alone. The third term is the player's expected value from finding the treasure in the current period together with the other player. The last term is the player's expected value from future periods. Note a player cannot observe where the other player unsuccessfully searched previously; that is, except for the area he himself has already searched, the rest of the island is observationally equivalent.

The SSMPE in the case when $\delta = 0.5$ and $x_1 = 0.74$ is described as follows.[24] The equilibrium search intensity is

$$I(x) = \begin{cases} x, & \text{if} \quad x \le 0.5, \\ x + \frac{1}{8} - \sqrt{\frac{1}{64} - \frac{1}{2}x + x^2}, & \text{if} \quad 0.5 < x \le 0.74; \end{cases}$$

and the value function is

$$V(x) = \begin{cases} (1 - 2x)/2, & \text{if} \quad 0 \le x \le 0.5, \\ I(x - x^2 - I/2)/x^2 + \frac{1}{2}(x - I)^2(\frac{1}{2} - x + I)/x^2, & \text{if} \quad 0.5 < x \le 0.74. \end{cases}$$

Figure 8 illustrates the equilibrium search intensity $I(x)$ and the value function $V(x)$ in the case with duplication. As shown in Figure 8a, the equilibrium search intensity is not piece-wise linear. The value function in Figure 8b is not monotonic in x; for example, $V(0.6) > V(0.5)$. The intuition for this result is as follows. For small island sizes, there is a dominant strategy for both players to search the whole island, despite duplication, because players share the treasure (receiving half of the prize) in the case of joint discovery. This holds whenever the payoff from this strategy is non-negative. Because of this, multi-period search is feasible only after the payoff from a one-period search strategy falls to zero. This naturally leads to non-monotonicity of the value function.

[24] See Appendix B for derivations.

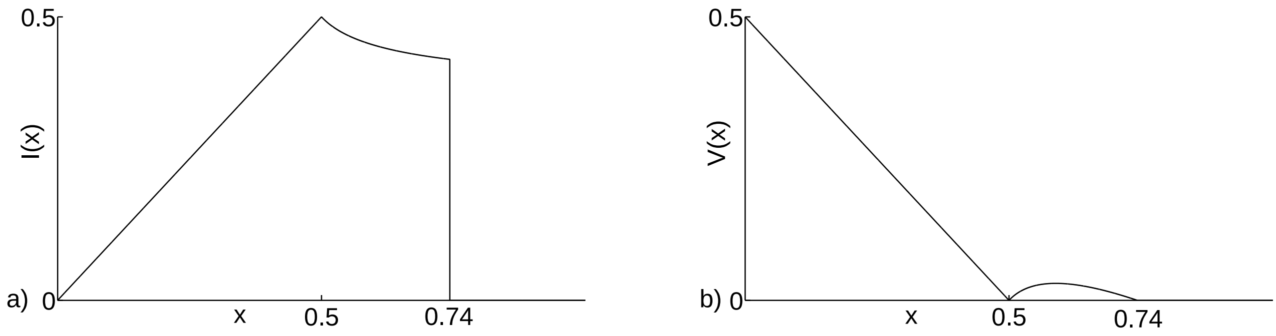

Figure 8: SSMPE for duopolists with duplication when $\delta = 0.5$ and $x_1 = 0.74$.

6 Discussion

The only dynamic R&D races that search literature has so far investigated rely on the memorylessness property of the exponential distribution. In contrast, in this paper we consider the case in which the success function is uniform. First, we identify two types of inefficiencies in the model: a tragedy of the commons (for small islands) and free riding (for large islands). These inefficiencies are manifested in the following way. When the race is far from completion firms underinvest. On the other hand, near the end of the race firms have incentives to overinvest. This differs from the literature, where only one of these inefficiencies is present for a single project. Second, expanding on the previous literature, we also investigate a multi-player case with three or more players, and demonstrate that in this case there is no monotonicity – players can be better off if the race is longer, even though such a race is more costly. From a policy making perspective, improving the conditions of firms, for example by increasing the size of the prize, may leave firms worse off.

Next, let us discuss some of the key aspects of the model. First of all, the assumption about one prize is very common in the theoretical R&D literature.[25] Second, we assume that there is an equal probability that the treasure will be located at any given point on the island. This assumption fits well the realistic case where there is no initial information on the possible location of the treasure. It also makes the analysis tractable. However, the case in which the distribution is not uniform has been discussed in the literature, see for example Fershtman and Rubinstein (1997), Choi and Gerlach (2014) and Chatterjee and Evans (2004). The expected outcome in our setting is that players start searching the parts of the island where the treasure is most likely to be found.[26] However, the possible complication here is that it is not clear when, if at all, the project will be completed. For instance, the optimal strategy could be to first search areas with a relatively high chance of success and, in the event of non-discovery, suspend the search because it becomes too costly and unprofitable.

Third, our solution concept is the SSMPE. There also exist asymmetric Markov

[25] See Reinganum (1989) for more detailed discussion about this assumption.
[26] Choi and Gerlach (2014) show that players start from the easiest project in R&D competition.

perfect equilibria, where in every period each player searches a different amount. Using a similar approach, one could consider the superior asymmetric Markov perfect equilibria for this game. We expect the same qualitative results hold as in our model. In addition, there exist other non-Markovian symmetric equilibria that rely on trigger strategies similar to those in Compte and Jehiel (2004). In the case of small islands, the only symmetric equilibrium that exists is the SSMPE. Players can always finish the search in one period, so trigger strategies are not useful in supporting cooperation. In contrast, in the case of large islands with a large number of potential investment periods, trigger strategies can improve cooperation. However, search of large islands must also be inefficient because, with enough unsuccessful search, the remaining unsearched area will eventually be like a small island. Consequently, efficient search can not be supported even with trigger strategies.

Finally, note the result that the value function is monotonic in the duopoly case is extremely fragile. In fact, all three extensions that we consider –uncertainty regarding the existence of the treasure, convex costs and duplication – lead to non-monotonicity of the value function when there are two players.

Appendix A

In this appendix we provide two alternative interpretations of the model presented in Section 3.

Private good provision with uncertain threshold

There are $n \geq 1$ players, who want to finance a private good as a team. The threshold of investment at which the good of value R becomes available is not known, but the players know that the threshold is distributed uniformly from 0 to some value x. In each period, all players simultaneously choose how much to contribute. If player i contributes I^i, his cost is cI^i. All players together contribute

$$J = I^1 + \ldots + I^n.$$

Note that since investing more than x has no benefit but is costly, it must be the case that $J \leq x$. If $J < x$, players have a J/x probability of financing the private good in the current period. If they are successful they share value R in proportion to their contributions; that is, player i gets I^i/J of the good (or gets R with probability I^i/J). If they are unsuccessful, the game is repeated next period with the threshold distributed uniformly from 0 to $x - J$, where future payoffs are discounted by the common discount factor δ. If $J = x$, players finance the private good in the current period. They share value R in proportion to their contributions, so that player i gets I^i/J of the good. One can see that the Bellman equation (1) describes this game. To our knowledge, our model is the first to consider a dynamic process of team investment in which the total cost to complete the project is uncertain.

Extraction of exhaustable common-pool resources (multi-player cake eating problem)

There are $n \geq 1$ firms, who each period choose simultaneously how much of a non-renewable resource to extract. The resource endowment is given by x and all firms have common access to the resource. If firm i extracts I^i in the current period, it derives net utility of $U_i = (1-x)I^i$ from selling the resource in the market. Firm i maximizes a discounted sum of utilities, where δ is the common discount factor. It is evident that the Bellman equation (5) describes this game. Note that when $x > 1$, net utility from selling the resource in the market is negative; this assumption diverges from the current literature. An example of this situation could be when mining companies need to extract a quantity of worthless material before they can access the valuable mineral deposit. Our setup leads to some interesting results, non-monotonicity of the value function for example, that is new to the existing literature on the economics of natural resources.

Appendix B

Proof of Lemma 1

To work with unit free variables introduce $\tilde{x} = cx/R$, $\tilde{I}^i = cI^i/R$ and $\tilde{I}^{-i} = cI^{-i}/R$ and substitute them into problem (1). This substitution results in

$$\bar{V}(\tilde{x}, 1, R) = \max_{\tilde{I}^i \in [0, \tilde{x} - \tilde{I}^{-i}]} \{-\tilde{I}^i R + \frac{\tilde{I}^i}{\tilde{x}} R + \delta \left(1 - \frac{\tilde{I}^i + \tilde{I}^{-i}}{\tilde{x}}\right) \bar{V}(\tilde{x} - \tilde{I}^i - \tilde{I}^{-i}, 1, R)\}, \quad (12)$$

where $\bar{V}(\tilde{x}, 1, R) = V(x, c, R)$. Next, substitute $\bar{\bar{V}}(\tilde{x}, 1, 1) = \bar{V}(\tilde{x}, 1, R)/R$ into equation (12) to derive

$$\bar{\bar{V}}(\tilde{x}, 1, 1) = \max_{\tilde{I}^i \in [0, \tilde{x} - \tilde{I}^{-i}]} \{-\tilde{I}^i + \frac{\tilde{I}^i}{\tilde{x}} + \delta \left(1 - \frac{\tilde{I}^i + \tilde{I}^{-i}}{\tilde{x}}\right) \bar{\bar{V}}(\tilde{x} - \tilde{I}^i - \tilde{I}^{-i}, 1, 1)\},$$

which coincides with problem (1) when $R = c = 1$. Consequently, $V(x, c, R)/R = V(cx/R, 1, 1)$ proving the lemma. □

Proof of Lemma 2

First, if $n = 1$, part (b) of the Lemma follows directly from the contraction mapping theorem (see for example Stokey, Lucas and Prescott (1989)). However this theorem is not useful for part (a) or when a multi-player game is considered. Consequently, we will provide a different proof, which will cover both single-player and multi-player games. For future reference note that given that the function $(1 - x)I$ in equation (5) is continuous, $\Psi(x)$ is also continuous.[27]

To prove (a), first consider the case when x is arbitrarily small. From equation (5) the marginal benefit of searching additional area in the first period is equal to the value that is

[27] See for example proposition 1 in Ericson and Pakes (1995) for a standard technique to prove a similar result.

arbitrarily close to 1, while the marginal benefit of postponing the search by one period is less than $\delta < 1$. Consequently, it is a dominant strategy for each player to finish the search in one period.

For the remaining values of x, suppose (a) is not satisfied. First, if $I_t = 0$ for some $t = \hat{t}$, the Markovian assumption implies investment remains zero for $\forall\ t > \hat{t}$. Hence, there are finitely many search periods. Assuming that the game lasts an infinite number of periods, this means that investment in some periods will take arbitrarily small amounts, i.e. for $\forall\ \varepsilon_0 > 0\ \exists\ \varepsilon \in (0, \varepsilon_0)$ and $\exists\ t \in N$ such that $I_t = \varepsilon$ and $n\varepsilon \ll x$. From (5) it follows:

$$\Psi(x) = (1-x)\varepsilon + \delta\Psi(x - n\varepsilon). \tag{13}$$

Expanding equation (13) in Taylor series around $\varepsilon = 0$ implies

$$\Psi(x) = -\frac{(1-x)\varepsilon}{1-\delta} + O(\varepsilon) = O(\varepsilon). \tag{14}$$

This means that $\Psi(x)$ must be arbitrarily close to zero. For values of $x < 1$ which are not arbitrarily small this is not possible, because each player will have incentive to deviate by choosing $I = x-(n-1)\varepsilon$ and generating $\Psi(x) = (1-x)(x-(n-1)\varepsilon) > 0$. When $x = 1$ a similar argument applies; in the first period a player has an incentive to deviate by making a positive contribution, for example $I = 0.5$. In this case, $\Psi(1) = \delta\Psi(0.5-(n-1)\varepsilon)$. Given that $\Psi(x)$ is not arbitrarily close to zero when $x < 1$, $\Psi(1)$ is also not arbitrarily close to zero.

When $x > 1$ a slightly different proof is used. For a given size of island $x_1 > 1$, each player makes a positive search $I > 0$ in the first period. Assuming that the game lasts an infinite number of periods, the investments have to take arbitrarily small values, i.e. for $\forall\ \varepsilon_0 > 0\ \exists\ \varepsilon \in (0, \varepsilon_0)$ and $\exists\ t \in N$ such that $I_t = \varepsilon$, where $x > 1$. From equation (14) it follows that when $I_t = \varepsilon$, $\Psi(x)$ is arbitrarily small. However, in the first period players make a positive search $I > 0$, which means they incur an immediate negative payoff of $(1 - x_1)I$. Subsequently, they make additional searches which add additional negative payoffs. Overall, this argument implies that $\Psi(x_1)$ is negative, which is a contradiction.

Thus, I_t cannot be arbitrarily close to zero, which means there exists $\zeta > 0$ such that $I_t > \zeta\ \forall\ t$. Consequently in any non-trivial SMPE the project has to be finished in a finite number of periods. This establishes part (a) of the Lemma.

Given that the project has to be finished in a finite number of periods, backward induction can be applied. Namely, we assume that $\Psi_0 \equiv 0$ and derive $\Psi_1 \equiv B\Psi_0$. This allows us to find all potential SMPE of the game if players could search for at most one period. Then we derive $\Psi_2 \equiv B\Psi_1$, which allows us to find all potential SMPE of the game if players could search for at most two periods. We continue this process until T is reached and find all potential SMPE of the game if players could search for at most T periods. Note that if $n = 1$ there always exists a unique equilibrium. If $n > 1$ SMPE is not necessarily unique. This establishes part (b) of the Lemma and concludes the proof. $\square$

Proof of Propositions 1, 2 and 3

Construction of Ψ_1 and V_1

Let us start from the end of the search process. What will be the value if players could only search for at most one period? Equation (5) transforms into

$$\Psi_1(x) = \max_{I \in [0, x - I^{-i}]} \{(1 - x)I\}. \tag{15}$$

It is evident that in the unique non-trivial SSMPE the equilibrium I can be described in the following way[28]:

$$I = x/n, \text{ if } x \leq 1.$$

If $x \leq 1$, then in SSMPE players search the whole island, $I^1 + ... + I^n = nI = x$. Consequently, the solution of (15) is

$$\Psi_1(x) = P_1(x), \text{ if } x \leq u_1 = 1; \tag{16}$$

where $u_1 = 1$ is the largest positive root of polynomial $P_1(x) = x(1 - x)/n$. For future reference note that

$$P_1(x) = \frac{a_1}{n}(1 - x)^2 + \frac{b_1}{n}(1 - x) + \frac{c_1}{n}, \tag{17}$$

where

$$a_1 = -1, \quad b_1 = 1, \quad c_1 = 0. \tag{18}$$

Define the value for each player (if the players can search the island for at most k periods) as $V_k(x) \equiv \Psi_k(x)/x$, for any $x \geq 0$. From the above definition, it follows that

$$V_1(x) = (1 - x)/n, \text{ if } x \leq u_1 = 1.$$

Construction of Ψ_2 and V_2

What will be the value if players can search the island for at most two periods? In general there could be three possibilities, depending on the island size. The first possibility is that the players search the whole island in just one period. Intuitively this happens for small values of x because it is too costly to wait for another period when the island is very small. The second possibility is that the players search the island for at most two periods. This happens for middle values of x. Finally, players can find search to be too costly. When the initial island is large (costs are very high), there is no non-trivial SMPE (mixed, assymetric or trivial equilibria are the only possibilities).

We have already considered the first possibility in the previous section. Now we analyze the situation when players plan to search for at most two periods. The first step is to construct $\Psi_2(x)$. Equation (5) in this case transforms into

$$\Psi_2(x) = \max_{I \in [0, x - I^{-i}]} \{(1 - x)I + \delta \Psi_1(x - I - I^{-i})\}. \tag{19}$$

[28] Note that if $x = 1$, then *any* $I \in [0, x/n]$ is optimal. We assume that players choose $I = x/n$ in this case.

The necessary condition for I to be optimal in the interior of $[0, x - I^{-i}]$ is

$$(1 - x) - \delta \Psi_1'(x - nI) = 0. \tag{20}$$

In order to continue the search for the second period, the remaining island size has to satisfy

$$0 \leq x - nI \leq u_1. \tag{21}$$

The sufficient condition for I to be optimal in the interior of $[0, x - I^{-i}]$ is satisfied because

$$\Psi_1''(x - nI) = 2a_1/n < 0.$$

The way to proceed is to construct the equilibrium with the help of condition (20), and then show that the derived equilibrium satisfies condition (21).

From expressions (20) and (16), it follows that

$$(1 - x) - \delta \left(\frac{1 - 2(x - nI)}{n} \right) = 0.$$

Consequently,

$$I = \frac{(2\delta - n)x + n - \delta}{2\delta n}. \tag{22}$$

Substituting (22) into equation (19), we obtain a *spline* of degree two on the interval $[0, u_2]$:

$$\Psi_2(x) = \begin{cases} \Psi_1(x), & \text{if} \quad 0 \leq x \leq t_1, \\ P_2(x), & \text{if} \quad t_1 < x \leq u_2; \end{cases} \tag{23}$$

where $u_2 > 0$ is the largest positive root of polynomial

$$P_2(x) = \frac{a_2}{n}(1 - x)^2 + \frac{b_2}{n}(1 - x) + \frac{c_2}{n}, \tag{24}$$

with

$$a_2 = -1 - s, \quad b_2 = \frac{1}{2}, \quad c_2 = \frac{\delta}{4}, \tag{25}$$

and

$$s = \frac{n(n - 2)}{4\delta}. \tag{26}$$

In order to find u_2, we need to solve the quadratic equation $P_2(u_2) = 0$. It is easy to check that

$$u_2 = 1 + \frac{\sqrt{4\delta(s + 1) + 1} - 1}{4(s + 1)}. \tag{27}$$

The point $x = t_1$ is the first knot of the spline. When $x = t_1$ players are indifferent between searching the island for two periods or for just one period:

$$\Psi_1(t_1) = \Psi_2(t_1). \tag{28}$$

From (17) and (24), we get[29]

$$t_1 = 1 - \frac{\delta}{n}. \tag{29}$$

All our calculations so far are valid for any $n \geq 1$. Consider parameter s now. From expression (26), it follows that

$$s \begin{cases} < 0, & \text{if} \quad n = 1, \\ = 0, & \text{if} \quad n = 2, \\ > 0, & \text{if} \quad n \geq 3. \end{cases} \tag{30}$$

Condition (30) characterizes three different types of behavior in SSMPE. There are three different cases, that are covered in three distinct theorems: $n = 1$ (a monopoly); $n = 2$ (a duopoly); and $n \geq 3$ (an oligopoly).

It is straightforward to check that the solution given by (23) satisfies condition (21) for any $x \in [t_1, u_2]$. Therefore, if the players can search the island for at most two periods, the SSMPE is a spline of degree one on the interval $[0, u_2]$ with one knot $x = t_1$:

$$I(x) = \begin{cases} \frac{x}{n}, & \text{if} \quad x \leq t_1, \\ \frac{(2\delta - n)x + n - \delta}{2\delta n}, & \text{if} \quad t_1 < x \leq u_2; \end{cases} \tag{31}$$

and the value function is

$$V_2(x) = \begin{cases} V_1(x), & \text{if} \quad x \leq t_1, \\ P_2(x)/x, & \text{if} \quad t_1 < x \leq u_2. \end{cases}$$

We can describe the construction of Ψ_k and V_k now.

Construction of Ψ_k and V_k

What will be the value if players can search the whole island for at most $k \geq 3$ periods? In general there could be $k + 1$ possibilities, depending on the island size x_1. The players can plan to search the island for at most $1, 2, ..., k$ periods in a non-trivial SSMPE. It is also possible that there is no non-trivial SMPE.

Let us construct $\Psi_k(x)$. Equation (5) in this case transforms into

$$\Psi_k(x) = \max_{I \in [0, x - I^{-i}]} \{(1 - x)I + \delta \Psi_{k-1}(x - I - I^{-i})\}. \tag{32}$$

A necessary condition for I to be the optimal value in the interior of $[0, x - I^{-i}]$ is

$$(1 - x) = \delta \Psi'_{k-1}(x - nI). \tag{33}$$

In order to continue search for the next period, the new value of x has to satisfy

$$t_{k-2} \leq x - nI \leq u_{k-1}, \tag{34}$$

[29] Condition (28) can have one or two solutions: one is given by (29), the other one exists only if $n > 2$ and is equal to $t_1 = 1 + \frac{\delta}{n-2}$. If $n = 1$, both solutions to (28) coincide. If $n = 2$, there is only one solution, which is given by (29). Finally if $n \geq 3$, it is easy to see that $\Psi_1(1 + \frac{\delta}{n-2}) < 0$, which means that only (29) is relevant.

where $t_0 = 0$. The sufficient condition for y to be the optimal value in the interior of $[0, x]$ is satisfied if

$$\Psi''_{k-1}(x - nI) < 0. \tag{35}$$

We will use condition (33) to find I, and then show that it satisfies conditions (34) and (35). Note that if function $\Psi_{k-1}(x)$ in (32) is a quadratic polynomial, then $\Psi_k(x) = B\Psi_{k-1}(x)$ has to be a quadratic polynomial as well. Since $P_1(x)$ and $P_2(x)$ are quadratic polynomials by (17) and (24), any $P_k(x)$ can be represented in the following form:

$$P_k(x) = \frac{a_k}{n}(1-x)^2 + \frac{b_k}{n}(1-x) + \frac{c_k}{n}, \quad k \geq 1. \tag{36}$$

From condition (33) and expression (36), it follows that

$$I = -\frac{(1-x)(n + 2\delta a_{k-1})}{2\delta n a_{k-1}} - \frac{b_{k-1}}{2na_{k-1}}. \tag{37}$$

Define the largest root of polynomial $P_k(x)$ as u_k, and that value of x such that players are indifferent between planning to search the area for k periods or for $k-1$ periods as knot t_{k-1}:

$$\Psi_{k-1}(t_{k-1}) = \Psi_k(t_{k-1}). \tag{38}$$

For the moment, let us assume that equation (38) has a unique solution. The uniqueness of the solution will be proved later in Lemma 3. Substituting (37) into equation (32), we obtain a *spline* of degree two on the interval $[0, u_k]$ with knots $t_1, ..., t_{k-1}$:

$$\Psi_k(x) = \begin{cases} \Psi_{k-1}(x), & \text{if} \quad 0 \leq x \leq t_{k-1}, \\ P_k(x), & \text{if} \quad t_{k-1} < x \leq u_k; \end{cases} \tag{39}$$

where $P_k(x)$ is defined in (36). Therefore, if players plan to search the island for at most k periods, then $I(x)$ is a spline of degree one on the interval $[0, u_k]$ with knots $t_1, ..., t_{k-1}$:

$$I(x) = \begin{cases} \frac{x}{n}, & \text{if} \quad x \leq t_1, \\ -\frac{(1-x)(n+2\delta a_1)}{2\delta n a_1} - \frac{b_1}{2na_1}, & \text{if} \quad t_1 < x \leq t_2, \\ \vdots & \\ -\frac{(1-x)(n+2\delta a_{k-1})}{2\delta n a_{k-1}} - \frac{b_{k-1}}{2na_{k-1}}, & \text{if} \quad t_{k-1} < x \leq u_k; \end{cases}$$

and the value function is

$$V_k(x) = \begin{cases} V_{k-1}(x), & \text{if} \quad 0 \leq x \leq t_{k-1}, \\ P_k(x)/x, & \text{if} \quad t_{k-1} < x \leq u_k. \end{cases}$$

Let us now find a_k, b_k, and c_k for any $k \geq 2$. Using (32), (36) and (37), we get the following system of difference equations:[30]

$$a_k = -1 + \frac{s}{a_{k-1}}, \quad b_k = -\frac{b_{k-1}}{2a_{k-1}}, \quad c_k = \delta\left(c_{k-1} - \frac{b^2_{k-1}}{4a_{k-1}}\right), \quad k \geq 2, \tag{40}$$

[30] Note that if $a_{k-1} = 0$, the optimal sequence includes $k-1$ steps only. This result follows from Lemma 3, and will be discussed at the end of the Lemma's proof.

where

$$s = \frac{n(n-2)}{4\delta}, \text{ and } a_1 = -1, \quad b_1 = 1, \quad c_1 = 0. \tag{41}$$

Let us consider cases $n = 1$, $n = 2$ and $n \geq 3$ separately. Theorem 1 characterizes the spline in (39) and the knots when $n = 1$.

Theorem 1. *When $n = 1$, system of difference equations (40) with initial conditions (41) has the following solution:*

$$a_k = -\frac{\sin(k+1)\varphi}{2v \sin k\varphi}, \ \ b_k = \frac{v^{k-1} \sin\varphi}{\sin k\varphi}, \ \ c_k = \frac{v^{2k-1} \sin(k-1)\varphi}{2 \sin k\varphi}, \tag{42}$$

$$t_k = 1 - v^k \cos k\varphi, \ \ u_k = 1 + \frac{v^k(\sin k\varphi - \sin\varphi)}{\sin(k+1)\varphi}, \ \ k \geq 1, \tag{43}$$

where $v = \sqrt{\delta}$, and $\varphi = \arccos v$.[31]

Proof. See below.

Theorem 2 characterizes the spline in (39) and the knots when $n = 2$.

Theorem 2. *When $n = 2$, system of difference equations (40) with initial conditions (41) has the following solution*[32]*:*

$$a_k = -1, \ \ b_k = \frac{1}{2^{k-1}}, \ \ c_k = \left(\frac{(4\delta)^{k-1} - 1}{4^{k-1}(4\delta - 1)}\right)\delta, \tag{44}$$

$$t_k = 1 - \frac{3\delta + (4\delta)^k(\delta - 1)}{2^k(4\delta - 1)}, \ \ u_k = 1 + \frac{1}{2^k}\left(\sqrt{\frac{1 - (4\delta)^k}{1 - 4\delta}} - 1\right), \quad k \geq 1. \tag{45}$$

Proof. See below.

Theorem 3 characterizes the spline in (39) and the knots when $n \geq 3$ players search for at most 2 periods in equilibrium.

Theorem 3. *When $n \geq 3$ and $k = 2$, system of difference equations (40) with initial conditions (41) has the following solution:*

$$a_1 = -1, \ \ a_2 = -1 - s, \ \ b_1 = 1, \ \ b_2 = \frac{1}{2}, \ \ c_1 = 0, \ \ c_2 = \frac{\delta}{4}, \ \ t_1 = 1 - \frac{\delta}{n}, \tag{46}$$

$$t_2 = 1 + \frac{-(1+2s) + \sqrt{(1+2s)^2 - 4s^2\delta(1 - 4(1-\delta)(1+s)}}{8s^2}, \ \ u_2 = 1 + \frac{\sqrt{4\delta(s+1)+1} - 1}{4(s+1)}. \tag{47}$$

[31]Note that $\sin k\varphi = 0$ implies $a_{k-1} = 0$ as in footnote 30.

[32]These expressions are presented in this concise form (rather than as sums) for expositional purposes. In particular, note that both numerators and denominators in c_k, t_k and u_k contain $(1 - 4\delta)$ term. When $\delta = 1/4$, these expressions are defined as their limits as $\delta \to 1/4$.

Proof. See below.

Finally, we demonstrate that conditions (34), (35) hold, and equation (38) has a unique solution.

Lemma 3. *For any value of x, (39) satisfies conditions (34) and (35). In addition, (38) has a unique solution.*

Proof. See below.

In order to complete the description of the SSMPE, we have to specify the maximum number of search periods for any value of x (i.e. we allow x to vary and find the maximum number of search periods). We will do that now.

Maximum number of search periods when $n = 1$ and $n = 2$

Here we answer the following question: what is the minimum number k such that $V(x) \equiv V_k(x)$? In other words, what is the maximum number of periods (the worst case scenario) in which the treasure will be found for certain?

One way to answer the above question is to write the condition that the largest positive root of the quadratic polynomial $P_k(x)$ coincides with the largest positive root of the quadratic polynomial $P_{k+1}(x)$. This condition gives a critical value of δ; for slightly larger values of δ, there is an additional search period.

This means that k is the smallest integer such that $\Psi_k(x) \equiv \Psi_{k+1}(x)$, or in other words $V(x) \equiv V_k(x)$. Specifically, when δ is at a critical value, k is the solution to the following equation:

$$u_k(\delta, n) = u_{k+1}(\delta, n). \tag{48}$$

Theorem 4 characterizes k for $n = 1$ and $n = 2$.

Theorem 4. *When $n = 1$, equation (48) has the following unique solution:*

$$k = \left\lceil \frac{\pi}{\arccos \sqrt{\delta}} - 1 \right\rceil. \tag{49}$$

When $n = 2$, equation (48) has the following unique solution:

$$k = \left\lceil \frac{\ln\left((\delta^2 + 2\delta)/(1-\delta)^2\right)}{\ln(4\delta)} \right\rceil. \tag{50}$$

Proof. See below.

Note that when $n = 2$ and $\delta = 1/4$ both numerator and denominator in k contain $\ln(4\delta)$ term, which means k is not well defined. The interpretation of this expression for k when $\delta = 1/4$ is its limit as $\delta \to 1/4$. This proves Propositions 1, 2 and 3. □

Proof of Theorem 1

Derivation of a_k, b_k and c_k

Let us show that when $n = 1$, equation (42) is the solution to the system of difference equations (40). Define

$$R_k = v^k \cdot \prod_{j=1}^{k} a_j \qquad k = 1, 2, \ldots \tag{51}$$

Using (40), one gets the following second-order difference equation

$$R_{k+1} = vR_k \cdot \left(-1 - \frac{1}{4\delta a_k}\right) = -vR_k - \frac{1}{4}R_{k-1} \quad k \geq 2. \tag{52}$$

The initial conditions are $R_0 = 1$ and $R_1 = -v$. The characteristic equation $4z^2 + 4vz + 1 = 0$ has two complex roots

$$z_1 = \frac{-v + ir}{2}, \quad z_2 = \frac{-v - ir}{2}, \qquad r = \sqrt{1 - v^2} > 0.$$

Denote $\varphi = \{\arg z_1 \in [0, \pi/2]\} = \arccos v$; then $z_{1,2} = -\frac{e^{\pm i\varphi}}{2}$. Further, write the solutions to equation (52) in the form $R_k = Az_1^{k+1} - Bz_2^{k+1}$, and use the initial conditions to get $A = B = -\frac{i}{\sin\varphi}$. Consequently

$$R_k = -\frac{i}{(-2)^{k+1}\sin\varphi}(e^{i(k+1)\varphi} - e^{-i(k+1)\varphi}) = -\frac{\sin[(k+1)\varphi]}{(-2)^k \sin\varphi}. \tag{53}$$

Apply (51) and (40) to get

$$a_k = \frac{R_k}{vR_{k-1}} = -\frac{\sin(k+1)\varphi}{2v\sin k\varphi}, \quad b_k = -\frac{b_{k-1}}{2a_{k-1}} = \frac{v^{k-1}\sin\varphi}{\sin k\varphi}, \tag{54}$$

$$\text{and } \; c_k = \delta\left[c_{k-1} - \frac{b_{k-1}^2}{4a_{k-1}}\right] = \frac{v^{2k-1}\sin(k-1)\varphi}{2\sin k\varphi}. \tag{55}$$

Derivation of t_k

To find t_k, one needs to solve the quadratic equation $P_k(t_k) = P_{k+1}(t_k)$; namely

$$(a_{k+1} - a_k)(1 - t_k)^2 + (b_{k+1} - b_k)(1 - t_k) + c_{k+1} - c_k = 0, \;\; k \geq 1. \tag{56}$$

Substitute a_k from (54) to derive

$$a_{k+1} - a_k = \frac{\sin(k+1)\varphi}{2v\sin k\varphi} - \frac{\sin(k+2)\varphi}{2v\sin(k+1)\varphi} = \frac{\sin^2(k+1)\varphi - \sin k\varphi\sin(k+2)\varphi}{2v\sin k\varphi\sin(k+1)\varphi} = \frac{\sin^2\varphi}{2v\sin k\varphi\sin(k+1)\varphi}. \tag{57}$$

Substitute b_k from (54), and note that $v = \cos\varphi$ to derive

$$b_{k+1} - b_k = \frac{v^k\sin\varphi}{\sin(k+1)\varphi} - \frac{v^{k-1}\sin\varphi}{\sin k\varphi} = \frac{2v^k\sin\varphi(\cos\varphi\sin k\varphi - \sin(k+1)\varphi)}{2v\sin k\varphi\sin(k+1)\varphi} = \frac{-2v^k\sin^2\varphi\cos k\varphi}{2v\sin k\varphi\sin(k+1)\varphi}. \tag{58}$$

Substitute c_k from (55), and note that $v = \cos\varphi$ to derive

$$c_{k+1} - c_k = \frac{v^{2k+1}\sin k\varphi}{2\sin(k+1)\varphi} - \frac{v^{2k-1}\sin(k-1)\varphi}{2\sin k\varphi} = \frac{v^{2k}(\cos^2\varphi\sin^2 k\varphi - \sin(k+1)\varphi\sin(k-1)\varphi)}{2v\sin k\varphi\sin(k+1)\varphi} = \frac{v^{2k}\sin^2\varphi\cos^2 k\varphi}{2v\sin k\varphi\sin(k+1)\varphi}. \quad (59)$$

Substitute the above relationships into (56), and cancel the non-zero common term $\frac{\sin^2\varphi}{2v\sin k\varphi\sin(k+1)\varphi}$ to derive

$$(1-t_k)^2 - 2v^k\cos k\varphi(1-t_k) + v^{2k}\cos^2 k\varphi = (1-t_k-v^k\cos k\varphi)^2 = 0.$$

Consequently,

$$t_k = 1 - v^k\cos k\varphi. \quad (60)$$

Note that both solutions to $P_k(t_k) = P_{k+1}(t_k)$ coincide, which means that the solution is unique.

Derivation of u_k

To find u_k, one needs to solve the quadratic equation $P_k(u_k) = 0$, namely

$$a_k(1-u_k)^2 + b_k(1-u_k) + c_k = 0, \;\; k \geq 1. \quad (61)$$

Substitute (54) and (55) into (61) to get

$$-(1-u_k)^2\sin(k+1)\varphi + 2(1-u_k)v^k\sin\varphi + v^{2k}\sin(k-1)\varphi.$$

Solving this quadratic equation results in

$$u_k = 1 + \frac{v^k(\sin k\varphi - \sin\varphi)}{\sin(k+1)\varphi}.$$

This concludes the proof. □

Proof of Theorem 2

Derivation of a_k, b_k and c_k

Let us show that when $n = 2$, formula (44) describes the solution to the system of difference equations (40). It is straightforward to derive $a_k = -1$ and $b_k = \frac{1}{2^{k-1}}$. The expression for c_k in (40) can be simplified to

$$c_k = \delta(c_{k-1} + 1/4^{k-1}). \quad (62)$$

Introduce a new variable $e_k = c_k 4^k$. Equation (62) transforms to

$$e_k = 4\delta(e_{k-1} + 1),$$

where $e_1 = 0$. The solution to this linear difference equation is $e_k = \frac{4\delta - (4\delta)^k}{1-4\delta}$. Substitute $c_k = e_k/4^k$ to derive

$$c_k = \frac{4\delta - (4\delta)^k}{(1-4\delta)4^k}.$$

Derivation of t_k

To find t_k, one needs to solve the quadratic equation $P_k(t_k) = P_{k+1}(t_k)$, namely

$$a_k(1-t_k)^2 + b_k(1-t_k) + c_k = a_{k+1}(1-t_k)^2 + b_{k+1}(1-t_k) + c_{k+1}, \quad k \geq 1.$$

From equation (44), $a_k = a_{k+1} = -1$; consequently,

$$t_k = 1 + \frac{c_{k+1} - c_k}{b_{k+1} - b_k}.$$

Substitute b_k and c_k from equation (44) to derive the following indifference points

$$t_k = 1 - \frac{3\delta + (4\delta)^k(\delta - 1)}{2^k(4\delta - 1)}.$$

Note that the solution to the quadratic equation $P_k(t_k) = P_{k+1}(t_k)$ is always unique.

Derivation of u_k

To find u_k, one needs to solve the quadratic equation $P_k(u_k) = 0$, namely

$$a_k(1-u_k)^2 + b_k(1-u_k) + c_k = 0, \quad k \geq 1.$$

Substituting $a_k = -1$ from equation (44) and solving the above quadratic equation gives

$$u_k = 1 + \frac{\sqrt{b_k^2 + 4c_k} - b_k}{2}. \tag{63}$$

Note that with the help of (44), one can simplify:

$$b_k^2 + 4c_k = \frac{(4\delta)^k - 1}{4^{k-1}(4\delta - 1)}. \tag{64}$$

Substitute equation (64) into equation (63) to get

$$u_k = 1 + \frac{\sqrt{\frac{1-(4\delta)^k}{1-4\delta}} - 1}{2^k}. \tag{65}$$

This concludes the proof. □

Proof of Theorem 3

Values in (46) are derived in (18), (25) and (29); u_2 is derived in (27).

Now let us find t_2. Applying (25) and (40) when $n \geq 3$, one can derive $a_2 = -1 - s$, $b_2 = \frac{1}{2}$, $c_2 = \frac{\delta}{4}$, $a_3 = -\frac{1+2s}{1+s}$, $b_3 = \frac{1}{4(1+s)}$ and $c_3 = \frac{\delta^2}{4} + \frac{\delta}{16(1+s)}$. It is easy to see that $a_3 - a_2 = 1 + s - \frac{1+2s}{1+s} = \frac{s^2}{1+s}$, $b_3 - b_2 = \frac{1}{4(1+s)} - \frac{1}{2}$, and finally $c_3 - c_2 = \frac{\delta}{16(1+s)} - \frac{\delta}{4}(1-\delta)$. To find t_2, one needs to solve the quadratic equation $P_2(t_2) = P_3(t_2)$, namely

$$a_2(1-t_2)^2 + b_2(1-t_2) + c_2 = a_3(1-t_2)^2 + b_3(1-t_2) + c_3,$$

which gives

$$t_2 = 1 + \frac{b_3 - b_2 + \sqrt{(b_3 - b_2)^2 - 4(a_3 - a_2)(c_3 - c_2)}}{2(a_3 - a_2)}. \tag{66}$$

Substitute values for $a_3 - a_2$, $b_3 - b_2$ and $c_3 - c_2$ into equation (66) and simplify to derive

$$t_2 = 1 + \frac{-(1+2s) + \sqrt{(1+2s)^2 - 4s^2\delta(1 - 4(1-\delta)(1+s))}}{8s^2}.$$

This concludes the proof. □

Proof of Lemma 3

Given that Proposition 3 only considers the case when players invest at most twice, the proof of the case when $n \geq 3$ is trivial. Consequently, the remainder of the proof deals with the cases when $n = 1$ and $n = 2$.

Define that part of x which player i does not search in the current period by $y = x - I^i$ and the part of x no player searches in the current period by $z = x - (I^1 + ... + I^n) = y - I^{-i}$.

Next, show that condition (34) is satisfied; that is, $t_{k-2} \leq z \leq u_{k-1}$. Let us prove the first part $z \leq u_{k-1}$ by contradiction, assuming that $z > u_{k-1}$. Refer to equation (32), which is written below:

$$\Psi_k(x) = \max_{I \in [0, x - I^{-i}]} \{(1-x)I + \delta\Psi_{k-1}(x - I - I^{-i})\}.$$

Given $x \geq z > u_{k-1} \geq \ldots \geq u_2 \geq u_1 = 1$, it follows that the first term $(1-x)I$ has to be non-positive. If $z > u_{k-1}$, then the second term $\delta\Psi_{k-1}(z)$ is negative. That means the whole expression on the right of equation (32) has to be negative. Obviously that could not be an optimal choice for a player because by choosing $y = x$, that is, by not searching, a player can get the value of zero. Consequently, there is a contradiction, and condition $z \leq u_{k-1}$ is proved.

Now let us show that $t_{k-2} \leq z$. When $n = 1$, we prove this inequality by contradiction, assuming that $z < t_{k-2}$. Note that by construction, when $z < t_{k-2}$, the following condition holds: $P_{k-1}(z) < P_{k-2}(z)$. That implies that instead of using the original k-period path (searching $x - y$ in the first period and making a further $k-1$ searches according to $P_{k-1}(z)$), the monopolist could use a $(k-1)$-period path (searching the same amount $x - y$ in the first period, and making a further $k-2$ searches according to $P_{k-2}(z)$), and increase the value. Refer to equation (32): both paths have the same first term, while the second term is larger for the $(k-1)$-period path. This implies that the k-period path does not improve the value in comparison with the optimal $(k-1)$-period path, which means that whenever $z < t_{k-2}$, the k-period path is not optimal. Condition $t_{k-2} \leq z$ is thus proved when $n = 1$.

When $n = 2$, let us prove the above condition directly. From equation (37) and Theorem 2, it follows that $z(x) = 1 - 2^{1-k} - \frac{1-x}{\delta}$. It is easy to see that $z(x)$ is a monotonically increasing function in x. Consequently, it is sufficient to prove the above condition for $x = t_{k-1}$. Substituting values derived in Theorem 2 and simplifying gives

$$\frac{3\delta + (4\delta)^{k-1}(\delta - 1)}{\delta(4\delta - 1)} \leq \frac{1 + 2\delta + 2(4\delta)^{k-2}(\delta - 1)}{4\delta - 1}.$$

Further simplifications result in $\frac{(4\delta)^{k-2}-1}{4\delta-1} \geq 0$, which is satisfied for any $k > 2$.

Let us show that condition (35) is satisfied; that is, $\Psi''_{k-1}(z) < 0$. From equation (39), it is clear that the sufficient condition for $\Psi''_{k-1}(z) < 0$ is that $P''_{i-1}(z) < 0 \; \forall \; i = 2, \ldots, k-1$. From equation (36), it is easy to see that the above condition is equivalent to $a_{i-1} < 0 \; \forall \; i = 2, \ldots, k-1$. For $n \geq 2$, it is straightforward to show that $s \geq 0$. From (40), one can see that a_k is a sum of a negative and a non-positive numbers; consequently, it has to be negative.

Now let us prove this condition for $n = 1$. Substitute t_k from (43) into (36) to get

$$P_k(t_k) = \frac{v^{2k-1}}{2n \sin k\varphi}(-\sin(k+1)\varphi \cos^2 k\varphi + 2\sin\varphi \cos k\varphi + \sin(k-1)\varphi). \tag{67}$$

Use the fact that

$$\sin(k+1)\varphi = \sin(k-1)\varphi + 2\sin\varphi\cos k\varphi$$

and substitute into (67) to derive

$$P_k(t_k) = \frac{-\delta^k a_k \sin^2 k\varphi}{n}. \tag{68}$$

Wherever the value function at t_k is positive, a_k has to be negative.

Next, for the proof that condition (38) has a unique solution, see the proofs of Theorems 1 and 2. In the case of $n = 1$, both solutions to (38) coincide; while in the case of $n = 2$, the second solution does not exist.

Finally, let us show that if $a_{k-1} = 0$, the optimal path includes $k-1$ steps only. A direct way of proving this result is to notice that if $a_{k-1} = 0$ in (68), then $P_{k-1}(t_{k-1}) = 0$, which means that the k-th step is unnecessary. □

Proof of Theorem 4

First, let us prove the result for $n = 1$. Equation (48) is equivalent to the condition $R_k = 0$ (R_k is defined in (51); see the last paragraph in the proof of Lemma 3). Apply (53) to get

$$\varphi_k \quad : \quad (k+1)\varphi = i\pi, \tag{69}$$

where $i \geq 1$ is some integer which can be different for different values of k, i.e. $i = i_k$. Let us prove by induction that $i_k = 1 \; \forall \; k \geq 2$. First, when $\delta = 1/4$, it follows that $\varphi = \pi/3$. Calculations show that $u_2 = t_2 = 9/8$, which means $k = 2$. Applying (69) results in $i_2 = 1$. Similarly, for $\delta < 1/4$ one can show that $t_2 > u_2$, which means $k = 2$ and $i_2 = 1$. Second, substitute $k' = k + 1$ in (69) to get

$$i_{k+1}\pi = (k+1)\varphi_{k+1} \leq (k+1)\varphi_k = \frac{k+1}{k}\pi < 2\pi.$$

We are using the inductive hypothesis that $i_k = 1$ and the fact that $\varphi_k = \arccos\sqrt{\delta_k}$ is monotonically decreasing in k. Given that i_{k+1} is an integer, it must be the case that $i_{k+1} = 1$. Substitute $i = 1$ in (69) to get $\varphi_k = \pi/(k+1)$, which means $k = \lceil \pi/\varphi - 1 \rceil$.

Next let us prove the result for $n = 2$. Substitute t_k and u_k from equation (45) into equation (48) to get

$$1+\frac{\sqrt{\frac{1-(4\delta)^k}{1-4\delta}}-1}{2^k}=1-\frac{3\delta+(4\delta)^k(\delta-1)}{2^k(4\delta-1)}.$$

Simplify the above expression to

$$\sqrt{\frac{1-(4\delta)^k}{1-4\delta}}-1=-\frac{3\delta+(4\delta)^k(\delta-1)}{(4\delta-1)}.$$

Further simplifications give

$$\sqrt{\frac{(4\delta)^k-1}{4\delta-1}}=\frac{((4\delta)^k-1)(1-\delta)}{(4\delta-1)},$$

and

$$\sqrt{\frac{(4\delta)^k-1}{4\delta-1}}(1-\delta)=1. \tag{70}$$

Square both sides of equation (70) to derive equation (50). This concludes the proof. □

Proof of Corollary 1

First, $I(x)$ presented in Proposition 1 is a piece-wise linear function. Second, let us prove that $I(x)$ is also a continuous function for $x \le u_m$. Given that $I(x)$ is piece-wise linear, discontinuities are only possible in the knots of the spline. Substitute $t_i = 1-\delta^{i/2}\cos i\varphi$ into $I(x)$, when the project is planned to be finished in $i \ge 1$ periods, to derive

$$I(t_i)=-\delta^{i/2}\cos i\varphi\left(1-\frac{\sin{(i-1)\varphi}}{\delta^{1/2}\sin i\varphi}\right)+\frac{\delta^{(i-1)/2}\sin\varphi}{\sin i\varphi}. \tag{71}$$

Next, substitute $t_i = 1-\delta^{i/2}\cos i\varphi$ into $I(x)$, when the project is planned to be finished in $i+1$ periods, to derive

$$I(t_i)=-\delta^{i/2}\cos i\varphi\left(1-\frac{\sin i\varphi}{\delta^{1/2}\sin{(i+1)\varphi}}\right)+\frac{\delta^{i/2}\sin\varphi}{\sin{(i+1)\varphi}}. \tag{72}$$

Use the fact that

$$\sin(i-1)\varphi\sin(i+1)\varphi=\sin^2 i\varphi-\sin^2\varphi, \tag{73}$$

to show that the expressions in (71) and (72) are equal. Note that there is a discontinuity at $x = u_m$.

Finally, to prove that $I(x)$ is a concave function it is sufficient to show that the slope of $I(x)$ is decreasing. Specifically, we need to show that

$$\left(1-\frac{\sin{(i-1)\varphi}}{\delta^{1/2}\sin i\varphi}\right)>\left(1-\frac{\sin i\varphi}{\delta^{1/2}\sin{(i+1)\varphi}}\right). \tag{74}$$

The above inequality follows from (73). □

Proof of Corollary 2

First, $V(x)$ presented in Proposition 1 is continuous. Remember that knot t_i is defined as an area size for which the monopolist is indifferent between planning to search the area for i periods or for $i+1$ periods.

Second, to prove that the value function is continuously differentiable for $x \leq u_m$, we need to show that derivatives of $V(x)$ coincide at knots t_i; that is,

$$\left(\frac{P_i(t_i)}{t_i}\right)' = a_i - \frac{a_i + b_i + c_i}{t_i^2} \sim a_{i+1} - \frac{a_{i+1} + b_{i+1} + c_{i+1}}{t_i^2} = \left(\frac{P_{i+1}(t_i)}{t_i}\right)'. \tag{75}$$

Simplify the above expression to

$$t_i^2 \sim 1 + \frac{b_3 - b_2 + c_3 - c_2}{a_3 - a_2}. \tag{76}$$

The substitution of values (57), (58), (59) and (60) shows the above expression (76) is satisfied with equality.

Finally, to show that the value function is monotonically decreasing we need to demonstrate that $\forall\ i \geq 2$ the following inequality holds:

$$V_i'(x) = \left(\frac{P_i(x)}{x}\right)' = a_i - \frac{a_i + b_i + c_i}{x^2} < 0. \tag{77}$$

Let us prove inequality (77) separately for $i \in [2, m-1]$ and for $i = m$. In the first case, from equation (68) it follows that $a_i < 0$. Consequently, it is sufficient to show that $a_i + b_i + c_i > 0$. From equation (56) and consecutive derivation of t_i, $P_i(x) > P_{i-1}(x)\ \forall\ x \neq t_i,\ \ i \geq 2$. This means, in particular, that $P_i(0) > P_{i-1}(0)$. Given that $P_1(0) = a_1 + b_1 + c_1 = 0$, it follows that $a_i + b_i + c_i > 0$.

On the other hand, when $i = m$, from equation (68) it follows that $a_i \geq 0$ and our previous argument does not hold. Note, however, that at $x = u_m$, inequality (77) has to hold because $V(x)$ crosses the horizontal axes from above. For values $x \in [t_{m-1}, u_m)$ it also has to hold, because for values of x smaller than u_m the expression on the left of (77) is even smaller. This proves the lemma. □

Proof of Corollary 3

First, $I(x)$ presented in Proposition 2 is a piece-wise linear function. Second, let us prove that $I(x)$ is not a continuous function. Given that $I(x)$ is piece-wise linear, discontinuities are only possible in the knots of the spline. Substitute $x = t_i$ into $I(x)$, when the project is planned to be finished in $i \geq 2$ periods, to derive

$$I(t_i) = \frac{2 - 2\delta + \delta/2^{i-2} - 2(1-\delta)t_i}{4\delta}. \tag{78}$$

Next, substitute $x = t_i$ into $I(x)$, when the project is planned to be finished in $i+1$ periods, to derive

$$I(t_i) = \frac{2 - 2\delta + \delta/2^{i-1} - 2(1-\delta)t_i}{4\delta}. \tag{79}$$

The expression in (78) is larger than the expression in (79); specifically, the difference is $\Delta I(t_i) = 1/2^{i+1}$.

Third, let us find the slope of $I(x)$. If the project is finished in one period then the slope is positive, $I'(x) = \frac{1}{2}$, otherwise it is negative, $I'(x) = \frac{\delta-1}{2\delta}$. This proves quasiconcavity and concludes the proof. □

Proof of Corollary 4

First, similar to Corollary 2, $V(x)$ presented in Proposition 2 has to be continuous. Second, to show that the value function is monotonically decreasing we need to show that $\forall\ i \geq 2$ inequality (77) holds. Substitution of $a_i = -1$ transforms (77) to

$$b_i + c_i + x^2 > 1. \tag{80}$$

For a given i, the left hand side is increasing in x, which means it is sufficient to prove the inequality for $x = t_{i-1}$. Substitute b_i, c_i and t_{i-1} from Proposition 2 to derive

$$\frac{1}{2^{i-1}} + \left(\frac{(4\delta)^{i-1} - 1}{4^{i-1}(4\delta - 1)}\right)\delta + \left(1 - \frac{3\delta + (4\delta)^{i-1}(\delta - 1)}{2^{i-1}(4\delta - 1)}\right)^2 > 1. \tag{81}$$

Let us prove inequality (81) separately for $i \in [2, 3]$ and for $i \in [4, d]$. When $i = 2$, the inequality transforms to $2 - 3\delta + \delta^2 > 0$, which is always satisfied for $\delta < 1$. Similarly when $i = 3$, the inequality transforms to $4 + 25\delta - 28\delta^2 + (3\delta - 4\delta^2)^2 > 0$, which is also always satisfied for $\delta < 1$.

From Proposition 2 when $d \geq 4$, it is easy to see that $\delta > 1/2$. Simplify inequality (81) to

$$(2^{i-1}-1)\left(2\frac{(4\delta)^{i-1} - 1}{(4\delta - 1)} - 1\right)(1-\delta) + \left(\frac{(4\delta)^{i-1} - 1}{(4\delta - 1)} + 1 - 2^{i-1}\right)\delta + \left(\frac{((4\delta)^{i-1} - 1)(\delta - 1)}{4\delta - 1}\right)^2 > 0.$$

The first and the third terms are positive for $i \geq 4$. The second term is increasing in δ. Substitute $\delta = 1/2$ to derive

$$\left(\frac{(4\delta)^{i-1} - 1}{(4\delta - 1)} + 1 - 2^{i-1}\right)\delta = (1 + 2 + \ldots + 2^{i-2} + 1 - 2^{i-1})\delta = 0. \tag{82}$$

This proves the lemma. □

Proof of Proposition 4

When $x = t_1$ players are indifferent between searching the island for two periods or for one period; that is, $V_1(t_1) = V_2(t_1)$. Let us show that $V_2'(t_1) > 0$ for $n \geq 3$, which means that there exists a value of x which is "slightly" larger than t_1 ($x > t_1$) such that $V(x) = V_2(x) > V(t_1)$:

$$V_2'(t_1) = \left(\frac{P_2(x)}{x}\right)'_{x=t_1} > 0. \tag{83}$$

Applying (24) and (29) transforms inequality (83) to

$$V_2'(t_1) = a_2 - \frac{a_2 + b_2 + c_2}{(1-\delta/n)^2} > 0. \tag{84}$$

With the help of (25) and (26), inequality (84) simplifies to

$$n^2(2n - 6 - 2\delta) + \delta(10n - 4\delta) > 0. \tag{85}$$

When $n \geq 4$, the above inequality always holds because both terms on the left are positive. When $n = 3$, inequality (85) simplifies to $\delta(12 - 4\delta) > 0$, which is also true. As a consequence, the proposition is proved. □

Proof of Proposition 5

Let us show that $\left(\frac{\partial V(x,c,R)}{\partial R}\right)_{x=t_1,c=1,R=1} < 0$ when $n \geq 4$.

Simplify the above expression to derive $\frac{\partial V(x,c,R)}{\partial R} = \frac{\partial\{RV(cx/R,1,1)\}}{\partial R} = cx\frac{\partial\{V(\tilde{x})/\tilde{x}\}}{\partial \tilde{x}}$. Consequently, from (83) it follows that all we need to show is that

$$\left(\frac{P_2(x)}{x^2}\right)'_{x=t_1} > 0. \tag{86}$$

Applying (24) and (29) transforms inequality (86) to

$$a_2 + b_2/2 - \frac{a_2 + b_2 + c_2}{1 - \delta/n} > 0. \tag{87}$$

With the help of (25) and (26), inequality (87) simplifies to $n > 3$. The proposition therefore is proved. □

Proof of Proposition 6

Let us prove this proposition in two steps. First, show that when $0 < \delta \leq 0.5$, knot $t_2 \geq 1$ for any $n \geq 1$, which implies $V(1) = V_2(1)$ for any n. Applying (43) when $n = 1$ gives $t_2 = 1 - v^2 \cos 2\varphi = 1 - \delta(2\cos^2\varphi - 1) = 1 + \delta(1 - 2\delta)$. It is easy to see that $t_2 \geq 1$ when $0 < \delta \leq 0.5$. Applying (45) when $n = 2$ gives $t_2 = 1 - \frac{3\delta + 16\delta^2(\delta-1)}{4(4\delta-1)} = 1 + \frac{3}{4}\delta - \delta^2$. It is easy to see that $t_2 \geq 1$ when $0 < \delta \leq \frac{3}{4}$. When $n \geq 3$ note that $a_3 - a_2 = 1 + s - \frac{1+2s}{1+s} = \frac{s^2}{1+s} > 0$, $b_3 - b_2 = \frac{1}{4(1+s)} - \frac{1}{2} < 0$, and finally $c_3 - c_2 = \frac{\delta}{16(1+s)} - \frac{\delta}{4}(1-\delta) = \frac{\delta(4\delta^2 - 3\delta - (1-\delta)n(n-2))}{4(4\delta + n(n-2))} < 0$ for $0 < \delta \leq \frac{3}{4}$. Applying (66), gives $t_2 > 1$ when $0 < \delta \leq \frac{3}{4}$.

Second, show that $nV_2(1)$ is the same for any n. Applying (36) gives $nV_2(1) = c_2 = \frac{\delta}{4}$. This proves the proposition. □

Proof of Proposition 7

Introduce $\tilde{x} = (x - \alpha)/(1 - \alpha)$, $\tilde{I}^i = I^i/(1 - \alpha)$ and $\tilde{I}^{-i} = I^{-i}/(1 - \alpha)$ and substitute them in problem (9). That results in

$$\bar{\Psi}(\tilde{x}) = \max_{\tilde{I}^i \in [0, \tilde{x} - \tilde{I}^{-i}]} \{(1 - \alpha)^2 (1 - \tilde{x})\tilde{I} + \delta \bar{\Psi}(\tilde{x} - \tilde{I}^i - \tilde{I}^{-i})\}, \tag{88}$$

where $\bar{\Psi}(\tilde{x}) = \Psi(x)$. Substitute $\bar{\bar{\Psi}}(\tilde{x}) = \bar{\Psi}(\tilde{x})/(1 - \alpha)^2$ and $\bar{\bar{\Psi}}(\tilde{x} - \tilde{I}^i - \tilde{I}^{-i}) = \bar{\Psi}(\tilde{x} - \tilde{I}^i - \tilde{I}^{-i})/(1 - \alpha)^2$ into equation (88) to derive

$$\bar{\bar{\Psi}}(\tilde{x}) = \max_{\tilde{I}^i \in [0, \tilde{x} - \tilde{I}^{-i}]} \{(1 - \tilde{x})\tilde{I} + \delta \bar{\bar{\Psi}}(\tilde{x} - \tilde{I}^i - \tilde{I}^{-i})\},$$

which coincide with problem (5). It follows that the substitutions $x' = \frac{x - \alpha}{1 - \alpha}$ and $\Psi' = \frac{\Psi}{(1 - \alpha)^2}$ allow to transform problem (9) into problem (5). This proves the lemma. □

Derivation of the example in section 5.2

Equation (10) in terms of $\Psi(x) = xV(x)$ can be written as

$$\Psi(x) = \max_{I \in [0, x - I^{-i}]} \{I - xI^2 + \delta \Psi(x - I - I^{-i})\}. \tag{89}$$

Furthermore, the solution has to satisfy $I^{-i} = I$. It is straightforward to prove the statement of Lemma 2 for the case of convex costs when $x \leq 1$; all the steps of the proof follow with very little amendment. The proof is available on request.

Let us start from the end of the search process. What will be the value if players could only search for at most one period? Equation (89) transforms into

$$\Psi_1(x) = \max_{I \in [0, x - I^{-i}]} \{I - xI^2\}. \tag{90}$$

If players can search the island for at most one period, then the only SSMPE is $(I(x), I(x))$, where

$$I(x) = x/2 \text{ if } x \leq 1,$$

$$\Psi_1(x) = x(2 - x^2)/4 \text{ if } x \leq 1,$$

and

$$V_1(x) = (2 - x^2)/4 \text{ if } x \leq 1.$$

Now let us construct $\Psi_2(x)$. Equation (89) in this case transforms into

$$\Psi_2(x) = \max_{I \in [0, x - I^{-i}]} \{I - xI^2 + \delta \Psi_1(x - I - I^{-i})\}. \tag{91}$$

The necessary condition for I to be optimal in the interior of $[0, x - I^{-i}]$ is

$$1 - 2xI - \delta \Psi_1'(x - I - I^{-i}) = 0. \tag{92}$$

The sufficient condition for I to be optimal in the interior of $[0, x - I^{-i}]$ is satisfied because

$$-2x + \delta\Psi_1''(x - I - I^{-i}) = -2x - \frac{6(x - I - I^{-i})}{4} < 0.$$

From expression (92), it follows that

$$1 - 2xI - 0.9\left(2 - 3(x - 2I)^2\right)/4 = 0.$$

Consequently,

$$I(x) = \begin{cases} x/2, & \text{if} \quad x \leq 0.74, \\ \frac{4.7 - \sqrt{14.8x^2 - 5.94}}{5.4}, & \text{if} \quad 0.74 < x \leq 1; \end{cases}$$

and the value function is

$$V(x) = \begin{cases} (2 - x^2)/4, & \text{if} \quad 0 \leq x \leq 0.74, \\ \left[4I(x)(1 - xI(x)) + 0.9(x - 2I(x))(2 - (x - 2I(x))^2\right]/(4x), & \text{if} \quad 0.74 < x \leq 1. \end{cases}$$

Note that it is quite challenging to derive $V_3(x)$, so we use numerical calculations to confirm that it is not optimal to search for more than two periods for any $x \leq 1$. Specifically, $t_2 \approx 1.08$. □

Derivation of the example in section 5.3

Equation (11) in terms of $\Psi(x) = xV(x)$ can be rewritten as

$$\Psi(x) = \max_{I \in [0,x]} \{I\left(1 - x - \frac{I^{-i}}{2x}\right) + \delta\left(1 - \frac{I^{-i}}{x}\right)\Psi(x - I)\}, \quad (93)$$

and the solution has to satisfy $I^{-i} = I$. As in the example above, it is straightforward to prove the statement of Lemma 2 amended for this problem; the proof is available on request.

Let us start from the end of the search process. What will be the value if players could only search for at most one period? Equation (93) transforms into

$$\Psi_1(x) = \max_{I \in [0,x]} \{I(1 - x - I^{-i}/(2x))\}. \quad (94)$$

If players can search the island for at most one period, the only SSMPE is $(I(x), I(x))$, where

$$I(x) = x \text{ if } x \leq 1/2,$$

$$\Psi_1(x) = x(1/2 - x) \text{ if } x \leq 1/2,$$

and

$$V_1(x) = 1/2 - x \text{ if } x \leq 1/2.$$

Now let us construct $\Psi_2(x)$. Equation (93) in this case transforms into

$$\Psi_2(x) = \max_{I \in [0,x]} \{I\left(1 - x - \frac{I^{-i}}{2x}\right) + \delta\left(1 - \frac{I^{-i}}{x}\right)\Psi_1(x - I)\}. \quad (95)$$

The necessary condition for I to be optimal in the interior of $[0, x]$ is

$$\left(1 - x - \frac{I^{-i}}{2x}\right) - \delta\left(1 - \frac{I^{-i}}{x}\right)\Psi_1'(x - I) = 0. \tag{96}$$

The sufficient condition for I to be optimal in the interior of $[0, x]$ is satisfied because

$$\Psi_1''(x - I) < 0.$$

From expression (96), it follows that

$$\left(1 - x - \frac{I}{2x}\right) - \frac{1}{2}\left(1 - \frac{I}{x}\right)\left(\frac{1}{2} - 2(x - I)\right) = 0.$$

Consequently,

$$I(x) = \begin{cases} x, & \text{if} \quad x \leq 0.5, \\ x + \frac{1}{8} - \sqrt{\frac{1}{64} - \frac{1}{2}x + x^2}, & \text{if} \quad 0.5 < x \leq 0.74; \end{cases}$$

and the value function is

$$V(x) = \begin{cases} (1 - 2x)/2, & \text{if} \quad 0 \leq x \leq 0.5, \\ I(x - x^2 - I/2)/x^2 + \frac{1}{2}(x - I)^2(\frac{1}{2} - x + I)/x^2, & \text{if} \quad 0.5 < x \leq 0.74. \end{cases}$$

As it is quite challenging to derive $V_3(x)$, we use numerical calculations to confirm that it is not optimal to search for more than two periods for any $x \leq 0.74$. □

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